MATHEMATICS
OF RADIO COMMUNICATIONS

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PREFACE

This book has been written to facilitate the study of radio, electronics, and allied subjects, which are broadly classified under the heading of communications. It is designed for those students who are unable to devote the required time to a preliminary study of mathematics but who, nevertheless, desire to pursue a serious study of communications.

The arrangement of the subject matter reflects the experience of several schools with accelerated programs. In a communications curriculum of limited time it is desirable that the mathematics be presented simultaneously with the communications studies, the subjects being interwoven so that each electricity topic is preceded by the requisite mathematics material. Accordingly, in this book certain mathematics subjects are divided—the minimum adequate introductory treatment being presented at one time with extensions and refinements following several chapters later. The sequence of topics is, thus, from a pure-mathematics view somewhat unusual.

It is suggested that the order of the first fifteen chapters be associated with the electrical studies in accordance with the following schedule, the mathematics in each case to be regarded as a prerequisite to the corresponding electrical work.

Electricity	CHAPTER	*Mathematics*
Elementary Direct Current Theory	1.	Arithmetic Operations
	2.	Simple Equations
Initial Laboratory Work	3.	Graphical Representation
	4.	Practical Computations
Direct Current Circuits	5.	Algebraic Operations
	6.	Exponents
	7.	Quadratic Equations. Square Root
	8.	Simultaneous Equations
Elementary Alternating Current Theory	9.	Trigonometric Functions
	10.	Radian Measure of Angles. Average Values
	11.	Rate of Change

Alternating Current Circuits	12. Solution of Triangles 13. Vectors 14. The Rotating Vector 15. Vector Forms

Electronics and general communications may occupy their natural position in the sequence of electrical studies, following the introductory basic circuit material.

The assistance of many individuals in the preparation of this work is gratefully acknowledged. Dr. James S. Webb and Dr. John F. Platt of the University of Minnesota read and criticized several of the early chapters of the manuscript. Dr. Dunham Jackson of the University of Minnesota proposed the novel technique of Section 10–5 for obtaining the half-wave sine average, and Dr. F. Carlin Weimer of Ohio State University offered many suggestions on the whole book.

T. J. W.

Columbus, Ohio
October, 1943

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Fundamental Processes

Laboratory Practice

Basic Circuit Mathematics

CHAPTER 1

ARITHMETIC OPERATIONS

1–1. Sliding Scales. The addition of two numbers may be conveniently accomplished with the use of two parallel scales as in Fig. 1–1.

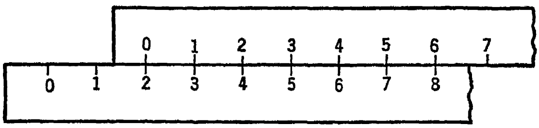


FIG. 1–1. Use of sliding scales for addition.

In the illustration of Fig. 1–1 the zero mark, or *index*, of the upper scale is set opposite the 2 on the lower scale. Now the addition of 3 to 2 involves finding what number occupies the third place beyond 2 in the scale of numbers. This is readily seen to be the 5 as the number on the lower scale which corresponds to 3 on the upper scale.

All the arithmetic processes of addition, subtraction, multiplication, and division may be performed with the aid of such a set of sliding scales. Thus, multiplication may be performed by continued addition: 1 times 2 is the same as the addition of 0 and 2 to yield 2; 2 times 2 involves the addition of 2 to the result of the preceding addition to yield 4; and 3 times 2 involves the addition of 2 to 4 to yield 6. In the foregoing the result of the *first* operation is 1 times 2; the *second* operation gives 2 times 2; and the *third* operation gives 3 times 2. One multiplier in each of these products is 2, and the other multiplier is the number of operations performed.

Subtraction is performed in the inverse manner of addition. Subtraction of 2 from 7 requires finding what number occupies the second place below 7 in the scale, and this process is performed by setting the sliding scale as shown in Fig. 1–2.

Division may be accomplished by a series of subtractions, the operations being the same as those for multiplication but in a reverse sequence. If 3 is the number of times that 6 must be subtracted from 18 to equal zero, then 18 divided by 6 equals 3.

Division of one number by a second number amounts to finding a third number such that the product of the second and third numbers is equal to the first. The problem $18 \div 6 = ?$ requires finding that number (3) which, when multiplied by 6, yields 18.

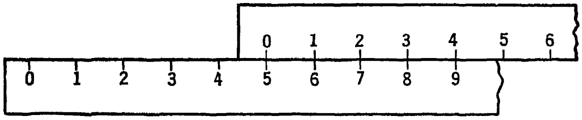


FIG. 1-2. Use of sliding scales for subtraction.

The division by zero of any number other than zero is undefined. The problem $18 \div 0 = ?$ requires finding that number which, when multiplied by zero, yields 18, and there is no number with this property.

By the introduction of fractions into the scale of numbers, division may be regarded as a product. Thus, $5 \div 8 = 5 \cdot \frac{1}{8}$.

1-2. Negative Numbers. In our scheme as developed thus far an operation, such as 3 minus 7, is undefined, since there is no number in the scale occurring in the seventh position before 3. However, the scale may be extended indefinitely with the use of *negative numbers*, and on this basis 3 minus 7 equals -4, as in Fig. 1-3.

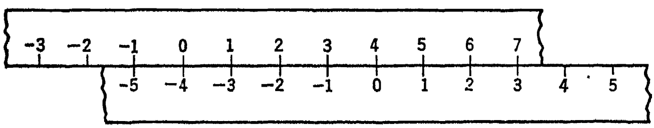


FIG. 1-3. Use of complete scales to extend the concept of subtraction.

In spite of the fact that a negative number so defined cannot be associated with a group of objects in the ordinary sense, the negative number has many practical applications. A common usage of negative numbers is as a measure of debit in contrast with the usage of positive numbers as a measure of credit.

All the numbers in the new complete scale lend themselves consistently to the familiar operations of arithmetic provided we introduce rules which are illustrated by the following examples:

$(-3) + (-2) = (-3) - 2$, that is, the second number before (-3) on the complete scale, which is (-5).

$(-3) - (-2) = (-3) + 2$, that is, the second number after (-3) on the complete scale, which is (-1).

$$(-4) \cdot (-2) = 4 \cdot 2 = 8.$$

$$4 \cdot (-2) = (-4) \cdot 2 = (-8).$$

$$\frac{(-4)}{2} = \left(-\left[\frac{4}{2}\right]\right) = (-2).$$

$$\frac{4}{(-2)} = \left(-\left[\frac{4}{2}\right]\right) = (-2).$$

$$\frac{(-4)}{(-2)} = \frac{4}{2} = 2.$$

Parentheses enclose negative numbers in the preceding equations so as to distinguish those minus signs which designate negative numbers from those minus signs which indicate subtraction. Frequently, when the context is clear, the parentheses are omitted. The rules above are stated in terms of specific numbers, but they apply to any and to all numbers of the complete scale.

Exercise 1-1

Perform the indicated operations:

1. $0 - (-6)$.
2. $2 + (-3)$.
3. $5 - 7$.
4. $5 - (-7)$.
5. $2 \cdot (-2)$.
6. $(-2) \cdot (9)$.
7. $(-5) \cdot (-3)$.
8. $(-4) \cdot (-4)$.
9. $-10 \div 5$.
10. $8 \div (-2)$.
11. $(-16) \div (-4)$.
12. $2 - 7 + 6$.
13. $\dfrac{4 \cdot (-3)}{(-2)}$.

1-3. Absolute Value. The *absolute value* or *numerical value* of a number is the value of the number without regard to sign. The absolute value of -6 is 6. The absolute value of any quantity is denoted by the use of a pair of vertical lines, thus:

$$|-3| = 3; \quad |3| = 3; \quad |-5 - 2| = |-7| = 7.$$

Exercise 1-2

Evaluate:

1. $|-4.2|$.
2. $|\frac{2}{3}|$.
3. $|5 - 4|$.
4. $|4 - 5|$.
5. $|-5 - 4|$.
6. $|-2 + 5 - 6|$.

1-4. Rational and Irrational Numbers. A *rational number* is a number which can be expressed as the quotient of two integers. An *irrational number* is one which cannot be so expressed. Examples of rational numbers are

$$5, \quad \frac{2}{3}, \quad 1\frac{1}{2}, \quad 8.3.$$

Examples of irrational numbers are

$$\pi = 3.14159 \cdots$$

$$\epsilon = 2.71828 \cdots$$

$$\sqrt{2} = 1.41421 \cdots$$

Any number which cannot be expressed as the quotient of two integers is not a part of the number system which is composed of positive and negative integers and fractions only. However, the system can be enlarged to accommodate irrational numbers, and the operations of addition, subtraction, multiplication, and division can be interpreted to include both rational and irrational numbers as a part of the same scheme. Rational and irrational numbers as a group are referred to as *real numbers*.

CHAPTER 2

SIMPLE EQUATIONS

2-1. Algebra. Scientific studies are facilitated by the use of letters to designate generalized quantities. Thus, the letter E may represent the magnitude of a voltage and the letter I may represent the magnitude of a current, wherein both E and I assume different values under different conditions.

Let us consider, for example, an experiment in which the voltage E across a particular electrical circuit is varied in steps from 0 to 10 volts, and corresponding meter readings of voltage across the circuit and current through the circuit are found to be as tabulated below.

E in volts	0	1	2	3	4	5	6	7	8	9	10
I in amperes	0	5	10	15	20	25	30	35	40	45	50

The observed relationship may be generalized by the equation $I = 5E$. We find it convenient here to speak, not of *one* specific value of the current or voltage, but of *any* specific value of the current or voltage.

We describe *one* particular value of a quantity by means of a number; thus, a current of 5 amperes; and we describe *any* value of a quantity by means of a letter; thus, a current of I amperes. A treatment of arithmetic operations and relationships with literal quantities constitutes the study of *algebra.*

2-2. Variables and Constants. A quantity which may take on a succession of values is called a *variable*; a quantity which has a single fixed value is called a *constant.* In the relation $I = 5E$, I and E are variables and 5 is a constant. It is conventional practice in mathematics to use in so far as is practicable, letters at the beginning of the alphabet — a b, c, d, e, f, — to designate constants, and letters at the end of the alpha bet — t, u, v, w, x, y, z — to designate variables. For example, unles otherwise specified, in the expression $ax + by + c$, the letters a, b, and

designate constants, that is, quantities which are fixed in magnitude throughout the discussion; and the letters x and y designate variables, that is, quantities which may assume different values during the course of the argument.

2-3. Special Characters. Because of the limited number of characters in the alphabet both upper case and lower case letters are used to represent various quantities, a different significance being attached to the upper and the lower case forms of the same letter. Sometimes boldface type or Old English type is employed to increase the number of available different representations, and recourse to Greek letters is common.

For reference the Greek alphabet is listed herewith. Those characters which are used most frequently in communications work are underscored for emphasis.

A	$\underline{\alpha}$	Alpha	I	ι	Iota	P	$\underline{\rho}$	Rho
B	$\underline{\beta}$	Beta	K	$\underline{\kappa}$	Kappa	$\underline{\Sigma}$	σ	Sigma
Γ	$\underline{\gamma}$	Gamma	Λ	$\underline{\lambda}$	Lambda	T	$\underline{\tau}$	Tau
$\underline{\Delta}$	$\underline{\delta}$	Delta	M	$\underline{\mu}$	Mu	Υ	υ	Upsilon
E	$\underline{\epsilon}$	Epsilon	N	$\underline{\nu}$	Nu	Φ	$\underline{\phi}$	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	$\underline{\psi}$	Psi
Θ	$\underline{\theta}$	Theta	Π	$\underline{\pi}$	Pi	$\underline{\Omega}$	$\underline{\omega}$	Omega

2-4. Meaning of Solution of an Equation. A set of values of the variables which fulfills the conditions of an equation is said to be a *solution* of the equation. A solution of the equation $I = 5E$ is $I = 5$, $E = 1$; another solution is $I = 0$, $E = 0$. There are, in fact, an indefinite number of solutions of the particular equation $I = 5E$. There is, however, one and only one solution of the equation $x + 1 = 4$. It is $x = 3$.

A set of values of the variables which constitutes a solution of an equation is said to *satisfy* the equation. To test whether or not a particular set of values of the variables satisfies an equation we substitute the values of the variables for the letters representing the variables in the equation. Thus, to test whether or not $I = 5$, $E = 1$ satisfies the equation $I = 5E$, we substitute 5 and 1, respectively, for I and E in the equation to obtain $5 = 5 \cdot 1$. The conclusion is that $I = 5$, $E = 1$

does satisfy the equation or, in other words, that $I = 5$, $E = 1$ is a solution of the equation.

Upon formally substituting 4 and 3, respectively, for I and E in the equation $I = 5E$, we obtain the absurdity $4 = 5 \cdot 3$. The conclusion is that $I = 4$, $E = 3$ does not satisfy the equation or, in other words, that $I = 4$, $E = 3$ is not a solution of the equation.

Incidentally, the fact that 4 is not equal to $5 \cdot 3$ may be expressed in abbreviated form by the notation "$4 \neq 5 \cdot 3$," where the symbol $\neq$ means: "is not equal to."

The value of 3 for x satisfies the equation $x + 1 = 4$ inasmuch as 1 added to 3 equals 4. Three is the only number to which 1 may be added to give 4. Hence, 3 is the only value of the variable x which satisfies the equation $x + 1 = 4$.

A solution of an equation is also spoken of as a *root* of the equation.

Exercise 2-1

A. Which of the following pairs of values of I and E satisfy the equation $I = 3E$?

1. $I = 0, E = 0$. 2. $I = 2, E = 2$. 3. $I = 1, E = -\frac{1}{3}$.

B. Which of the following values of y satisfy the equation $y - 7 = 2$?

1. $y = 5$. 2. $y = 0$. 3. $y = 9$.

C. Obtain solutions for the following equations:

1. $6 = \frac{1}{2}x$. 2. $5z = 10$. 3. $x + 1 = 3$.

2-5. Fundamental Law of Equalities. We shall consider as *equivalent equations* any equations which have the same solutions. The process of finding the solution or solutions of an equation may usually be simplified by the use of the axiomatic fundamental law of equalities: If both sides of an equation are increased, decreased, multiplied, or divided by the same quantity (division by zero excluded), the result is an equivalent equation.

Example 1. Solve the equation $x + 1 = 3$.
Subtracting 1 from both sides, we obtain

$$x + 1 - 1 = 3 - 1,$$

or

$$x = 2.$$

That $x = 2$ is a solution of the given equation, $x + 1 = 3$, may be verified by substituting 2 for x in the given equation. Thus: $2 + 1 = 3$.

Example 2. Solve the equation $-8x = 40$.

Dividing both sides by -8, we obtain

$$\frac{-8x}{-8} = \frac{40}{-8},$$

or

$$x = -5.$$

That $x = -5$ is a solution of the given equation, $-8x = 40$, may be verified by substituting -5 for x in the given equation. Thus: $-8 \cdot (-5) = 40$.

Exercise 2-2

A. Solve the following equations, and verify the result in each case:

1. $x - 5 = 0$.
2. $\frac{x}{2} = 2\frac{1}{2}$.
3. $7x = 0$.
4. $x + 10.5 = 1.3$.

B. Show that if $P + Q = A$ and $P - Q = B$, then $P = \frac{A + B}{2}$ and $Q = \frac{A - B}{2}$.

(Hint: $P - Q$ equals B. Hence, as regards the equation, $P + Q = A$, to the left side we may add $P - Q$, and to the right side we may add B. The result is the new equation, $P + Q + P - Q = A + B$.)

2-6. Proportionality. In the relation $I = \frac{E}{R}$ we say that I is *directly proportional to E*, or, simply, that I is *proportional to E*; and we say that I is *inversely proportional to R*.

In each of the following equations I is proportional to E:

$$I = 3E,$$

$$I = -\tfrac{1}{2}E,$$

$$I = 500E.$$

In the first equation, where I is equal to 3 times E, 3 is spoken of as the *constant of proportionality*. The constant of proportionality is $-\frac{1}{2}$ in the second equation and 500 in the third.

If we wish to express the general relationship exhibited in all three of the above equations, that is, I is proportional to E, we write

$$I \propto E,$$

where the symbol $\propto$ means: "is proportional to."

To express a general inverse proportion between two variables I and R, we may write

$$I \propto \frac{1}{R}.$$

If $I = \frac{1}{R}$, we say that I is the *reciprocal* of R.

Exercise 2-3

A. Indicate in which of the following relations y is proportional to x. In each such case specify the constant of proportionality.

1. $y = x + 2.$
2. $y = x.$
3. $y = -x.$
4. $y = \frac{1}{x}$
5. $y = 5x.$
6. $y = 2.$

B. Show that:

1. If $y \propto x$, then $x \propto y$. (Hint: Write $y \propto x$ as $y = kx$.)
2. If $y \propto \frac{1}{x}$, then $x \propto \frac{1}{y}$.
3. If y is the reciprocal of x, then x is the reciprocal of y.

CHAPTER 3

GRAPHICAL REPRESENTATION

3–1. Graphs. Certain engineering relationships may be portrayed through charts or diagrams. The fact that a force on a charged body varies inversely as the square of the distance of the body from a certain point may be represented either in algebraic symbols, by an equation of the form $F = \frac{q}{r^2}$ (where q is a constant), or in pictorial representation, by a curved line the height of which is at each point inversely proportional to the square of its distance from some reference line, as in Fig. 3–1.

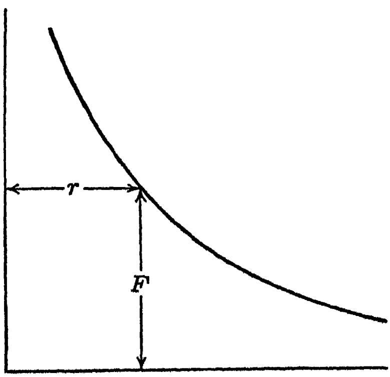


FIG. 3–1. Graph designed to show variation of F according to the relation $F = \frac{q}{r^2}$.

A diagram such as that of Fig. 3–1 is referred to as a *graph*. Of the algebraic and graphical types of portrayal of engineering relationships, the former is frequently more precise, but the latter is often more lucid. Both methods are employed in practice, the choice in any instance being dictated by practicability and by convenience.

3–2. Coordinates. Numbers which designate the position of a point in a graph are called the *coordinates* of that point. In Fig. 3–2 point P is 3 units to the right of, and 2 units above, the reference point O. Three and 2 are the coordinates of P. In specifying a point by coordinates it is conventional to state the horizontal distance from the reference point first. Thus, the coordinates of P are given in the order 3,2; and the point P is uniquely placed on the graph by the designation (3,2). In like fashion the coordinates of Q are $(-2,5)$; the coordinates of R are $(-4,-2)$; and the coordinates of S are $(3,-3)$.

The locating of points on a graph, when their coordinates are given, is facilitated by the use of *graph paper* which is ruled with sets of vertical and horizontal lines.

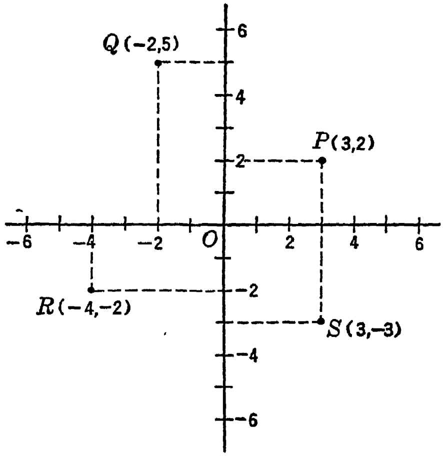


FIG. 3–2. Designation of points by coordinates.

Exercise 3–1

Locate the following points on a graph:

1. (5,2).
2. (0,2).
3. (0,0).
4. (−8, −4).
5. (−6,0).
6. (3, −2).

3–3. Coordinate System. In Fig. 3–2 the reference point O is called *the origin* of the coordinate system. The horizontal distance of a given point from the origin is referred to as the *x-coordinate*, or *abscissa*, of the point; and the vertical distance from the origin is referred to as the *y-coordinate*, or *ordinate*, of the point. Thus a general point P has coordinates (x,y). The horizontal and vertical reference lines through the origin are called, respectively, the *x-axis*, or *axis of abscissas*, and the *y-axis*, or *axis of ordinates*.

The four regions which are bounded by the coordinate axes are referred to as the *quadrants*. The quadrants are numbered in counterclockwise order beginning with the upper right quadrant. In Fig. 3–2 the point P is in the *first quadrant*; the point Q is in the *second quadrant*; the point R is in the *third quadrant*; and the point S is in the *fourth quadrant*.

The length of the line segment joining the origin to a point is called the *radius vector* of the point. The radius vector is always taken to be positive. In Fig. 3–2 the radius vector of P is $\sqrt{3^2 + 2^2} = \sqrt{13}$; the radius vector of Q is $\sqrt{2^2 + 5^2} = \sqrt{29}$; the radius vector of R is $\sqrt{4^2 + 2^2} = \sqrt{20}$; and the radius vector of S is $\sqrt{3^2 + 3^2} = \sqrt{18}$.

Exercise 3–2

Indicate in which quadrant each of the following points is located, and determine the radius vector in each case:

1. (2,17).
2. (17,2).
3. (−1,3).
4. (−6,−5).
5. (10,−7).
6. (7,−10).

3–4. The Graph of an Equation. The series of points (−10,2), (−5,2), (0,2), (8,2), (11,2) is located in Fig. 3–3. All of the points lie on a horizontal line which is 2 units above the x-axis. It is readily seen that the y-coordinate of every point which lies on this line* is 2 and, further,

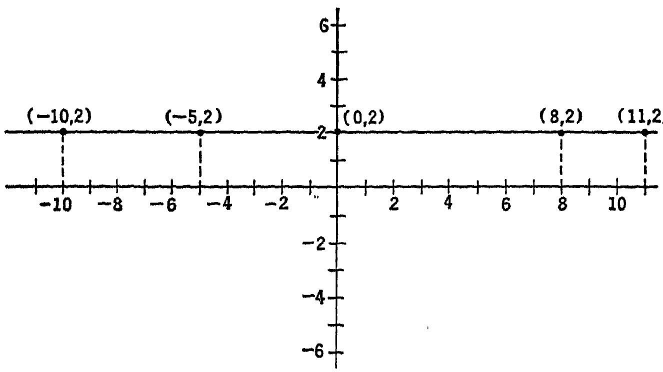


FIG. 3–3. Graph of $y = 2$.

that the y-coordinate of no point which lies off this line is 2. In other words, the line includes every point which is of ordinate 2 and no point which is not of ordinate 2. The line is said to be the *graph*, or *curve*, of all points whose y-coordinates satisfy the relation $y = 2$; and the equation $y = 2$ is said to be the *equation* of the line.

*The word "line" by itself is used here and in the remainder of this text in the sense of a straight line.

The equation $x = 3$ represents a series of points each having an x-coordinate of 3. Thus, the equation $x = 3$ has as its graph the vertical line which is 3 units to the right of the y-axis (Fig. 3-4).

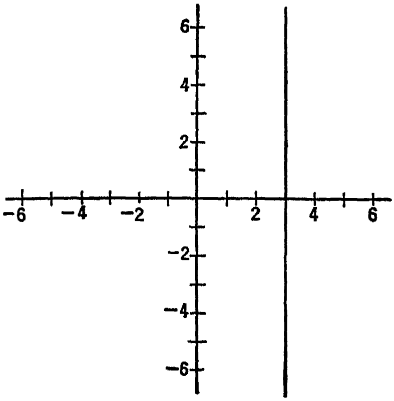


FIG. 3-4. Graph of $x = 3$.

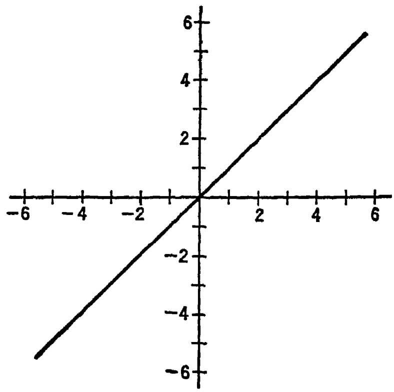


FIG. 3-5. Graph of $x = y$.

The equation $x = y$ is satisfied by every point whose x- and y-coordinates are equal. The graph of this equation is, then, the line formed by the series of points $(-2,-2)$, $(-1,-1)$, $(0,0)$, $(1,1)$, $(2,2)$, etc. This is the line which passes up and to the right through the origin, making an angle of 45° with each of the coordinate axes (Fig. 3-5).

Exercise 3-3

Draw the graphs of the following equations:

1. $x = 0$. 2. $y = -2$. 3. $y = -x$.

3-5. Graph Plotting. When an equation is given which states a relation between x and y, the corresponding graph can be drawn by *plotting* a number of points as demonstrated in the following example.

Example. Draw the graph of $y = \dfrac{x^2}{2}$.

The point $(0,0)$ is found to lie on the graph, since substitution of 0 for x in the equation $y = \dfrac{x^2}{2}$ yields accordingly $y = 0$. In like manner pairs of corresponding values of x and y for a number of points on the graph may be computed. Some of these are tabulated below. These points are plotted in

Fig. 3-6. The graph is completed by *interpolating* (filling in) points between the plotted points to make a smooth curve.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	12.5	8	4.5	2	0.5	0	0.5	2	4.5	8	12.5

We may speak of the graph of Fig. 3-6 as a plot of y *against* x, or of y *versus* x, for the relation $y = \frac{x^2}{2}$. The graph of Fig. 3-8 in Sec. 3-8 is labelled a "graph of plate current versus plate potential."

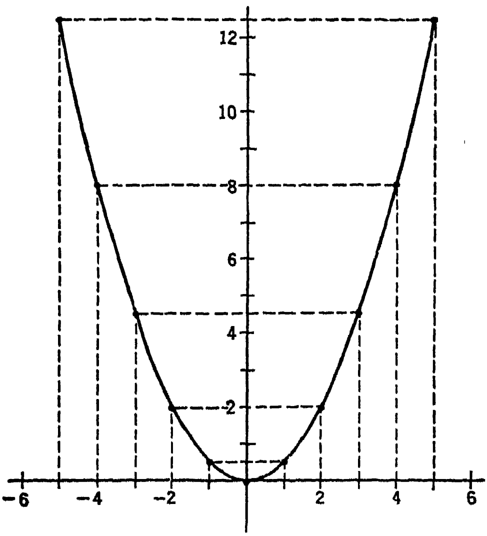


FIG. 3-6. Graph of $y = \frac{x^2}{2}$.

Exercise 3-4

A. Plot graphs for positive values of x for the following equations:

1. $y = 2x + 3$.
2. $y = 2x - 3$.
3. $y = \frac{1}{x}$.
4. $y = \frac{1}{x^2}$.

B. Using the same set of coordinate axes, plot curves of X_L versus f, $-X_C$ versus f, and X versus f, where

$$X_L = 2\pi f L,$$

$$X_C = \frac{1}{2\pi f C},$$

and

$$X = X_L - X_C.$$

Take $L = 10$ and $C = 0.00001$, and plot the curves over the interval from $f = 0$ to $f = 50$. Plot f along the x-axis.

3-6. Intercepts. The *x-intercept* of a curve is the abscissa which corresponds to $y = 0$; and the *y-intercept* of a curve is the ordinate which corresponds to $x = 0$. To find the x-intercept of a curve from the equation of the curve we set y equal to zero in the equation of the curve and solve for x. To find the y-intercept we set x equal to zero and solve for y.

Example. Without recourse to the plotted curve, obtain the x- and y-intercepts of the curve whose equation is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Setting y equal to zero, we obtain

$$\frac{x^2}{25} = 1 \quad \text{or} \quad x^2 = 25,$$

from which we find the x-intercepts as 5 and -5. Setting x equal to zero, we obtain

$$\frac{y^2}{9} = 1 \quad \text{or} \quad y^2 = 9,$$

from which we find the y-intercepts as 3 and -3. The plotted curve is shown in Fig. 3-7.

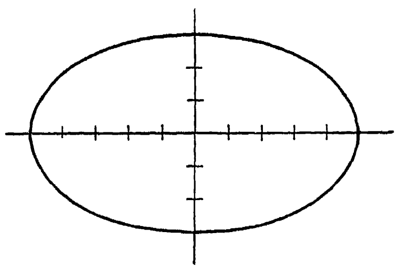


FIG. 3-7. Graph of $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

Exercise 3-5

Without plotting, obtain the x- and y-intercepts of the graphs of the following equations:

1. $2x + 3y + 7 = 0.$
2. $5x - 2y + 1 = 0.$
3. $y^2 = 2x + 9.$
4. $x^2 + 4y^2 = 64.$

3-7. Linear Relationship. It will be observed from the graphs which were drawn in the preceding sections that a line was associated in each case with an equation of the type $Ax + By + C = 0$. For example, a line resulted on plotting points corresponding to $x = y$ in Fig. 3-5. Now, the equation $x = y$, with y subtracted from each side, becomes $x - y = 0$; and the latter (equivalent) equation, $x - y = 0$, is recognized as an equation of the general type $Ax + By + C = 0$, wherein $A = 1$, $B = -1$, and $C = 0$. Again, the plotting of points corresponding to $y = 2x + 3$ in Problem 1 of Exercise 3-4 should have been found likewise to result in a line. The equation $y = 2x + 3$, with y subtracted from each side, yields the equivalent equation $0 = 2x - y + 3$; and the equation $0 = 2x - y + 3$ is recognized as a special case of the equation $Ax + By + C = 0$ wherein $A = 2$, $B = -1$, and $C = 3$. It will be shown in Sec. 20-5 that the plotting of points associated with any equation of the type $Ax + By + C = 0$ results in a line and, further, that every line is represented by an equation of the type $Ax + By + C = 0$. For this reason it is customary to refer to an equation of the type $Ax + By + C = 0$ as a *linear equation*. (An equation of any type other than linear is referred to as *non-linear*.)

For the present we can take advantage of the relationship pointed out in the preceding paragraph in the plotting of graphs which are associated with linear equations. Once an equation is identified as of the linear type, the property of linearity of the graph is established; and it then is necessary to plot only any two points which lie on the graph, the remainder of the graph being completed by drawing a line through the two plotted points. In the selection of two points for the construction of a line from its equation, it will usually simplify the computations to choose the two intercepts.

Example. Draw the graph of $x + 4y - 12 = 0$.

This equation, being of the form $Ax + By + C = 0$, represents a line. On substituting zero for y in the equation $x + 4y - 12 = 0$, we obtain $x - 12 = 0$, or $x = 12$. Thus, the x-intercept is 12. On substituting zero for x in the equation $x + 4y - 12 = 0$, we obtain $4y - 12 = 0$, or $y = 3$. Thus, the y-intercept is 3. The graph of the line is constructed in Fig. 3-8 by drawing a line through the points (12,0) and (0,3).

In the use of intercepts in the plotting of lines, exceptional cases arise for lines which have only one intercept, as a line which is parallel to one of the axes or a line which passes through the origin. Further, a line which passes near the origin is apt to have its intercepts inconveniently close together, so

that the direction of the line is not accurately determined from the two intercepts. In any such case, however, it is an easy matter to choose a satisfactory additional point, if desired, to complete the determination of the graph. A

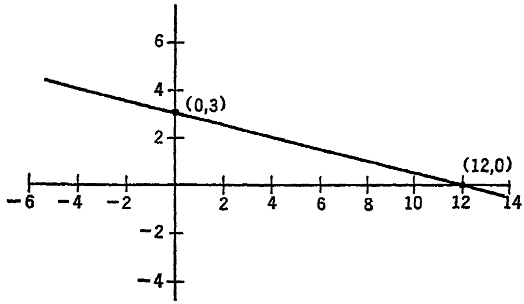


FIG. 3–8. Two point construction of graph of $x + 4y - 12 = 0$.

line which is parallel to one of the coordinate axes is readily identified from its equation through the fact that its equation is equivalent to one of the form $x = k$ or $y = l$, where k and l are any two constants. (Compare Sec. 3–4.)

Exercise 3–6

A. Indicate which of the following equations represent lines:

1. $y = \dfrac{3x - 2}{4}$. (Hint: First multiply both sides of the equation by 4. Then subtract $4y$ from both sides.)

2. $\dfrac{y}{3} = -2x$.

3. $y = \sqrt{x}$.

4. $\dfrac{3}{y} = 7x$.

5. $3 + 2y = 0$.

6. $xy + 2 = 0$.

7. $\dfrac{y}{2} = x + 5$.

B. Construct graphs corresponding to the linear equations in part A above.

C. Representing the variable E by distances measured along the horizontal axis, and the variable I by distances measured along the vertical axis, draw

lines of $I = \frac{E}{R}$ for $R = \frac{1}{2}$, $R = 1$, $R = 2$, and $R = 10$. Use the same set of axes for all four lines.

D. Starting with $Ax + By + C = 0$ as the general equation of a line, show that $Ax + By = 0$ is the general equation of a line which passes through the origin. (The equation must be satisfied by (0,0), the coordinates of the origin.) Show that $I = \frac{E}{R}$, where I and E are variables and R is a constant may be regarded as a special case of the equation $Ax + By = 0$

E. Find the equation of the line which passes through the origin and through the point (1,2).

3-8. Graph Plotting from Experimental Data. A laboratory test conducted on a type 1H5 vacuum tube yields the following sets of corresponding values of plate potential and plate current:

Plate volts	50	82	100	120	152	168
Plate microamperes	20	110	195	310	520	660

These data are plotted in Fig. 3-9. The graph includes all the information of the tabulated data, but the graph in addition *interpolates* (fills in) between the plotted points to present at a glance a complete picture of the plate current-plate potential relationship.

The graph of Fig. 3-9 illustrates certain conventional features of graphs of experimental data:

1. The experimentally observed values are represented by small circles, a smooth curve joining one circle to the next. The curve touches, but does not penetrate into, each circle.
2. The graph carries a title.
3. Scale values of abscissas and ordinates are indicated.
4. That quantity which is represented by the abscissas is designated, together with the unit in which it is measured, by a legend along the x-axis. That quantity which is represented by the ordinates is designated, together with the unit in which it is measured, by a legend along the y-axis.
5. The inscription along the y-axis is in such a direction as to be legible when viewed from the right side.

Deviations from the practices mentioned above are occasionally

encountered. For example, it is customary to use straight lines instead of smooth curves to join experimental points whenever it is desired to indicate specifically that the course of the graph between the plotted

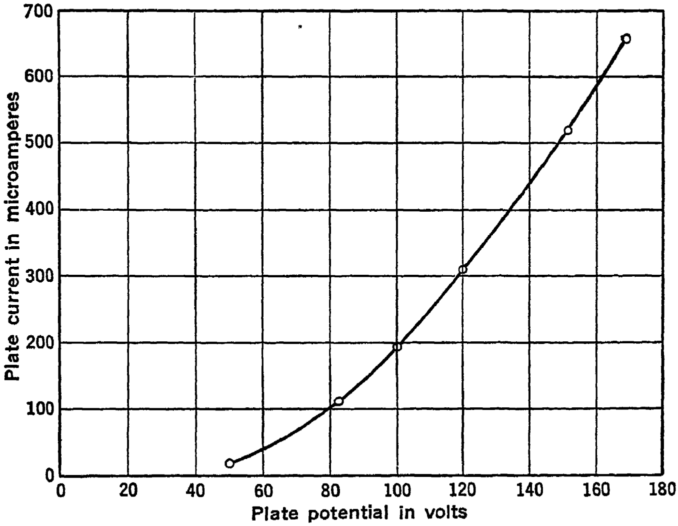


FIG. 3–9.

points is uncertain. Thus, electrical meter scale corrections are often represented by zigzag graphs wherein straight lines join those points which correspond to actual calibration data.

Whenever several graphs are drawn on one sheet, the graphs should be distinguished from one another by some convention, and a classification key should be included on the graph. The use of various colors to distinguish individual curves is desirable, but where this is not feasible, various types of dotted or dashed curves may be employed. The plotted points of the individual curves may be distinguished through the use of such devices as circles, crosses, squares, triangles, and diamonds. The technique of plotting more than one experimental curve on the same sheet is illustrated by Fig. 3–10, which is reproduced from an article by H. W. Wells on the solar effect on radio, in the Proceedings of the Institute of Radio Engineers for April, 1943.

In drawing curves representing mathematical equations, as distinct from curves of experimental data, it is generally not essential to indicate the plotted points.

3-9. Dependent and Independent Variables. The relation between two quantities is frequently such that one may be regarded as a cause and

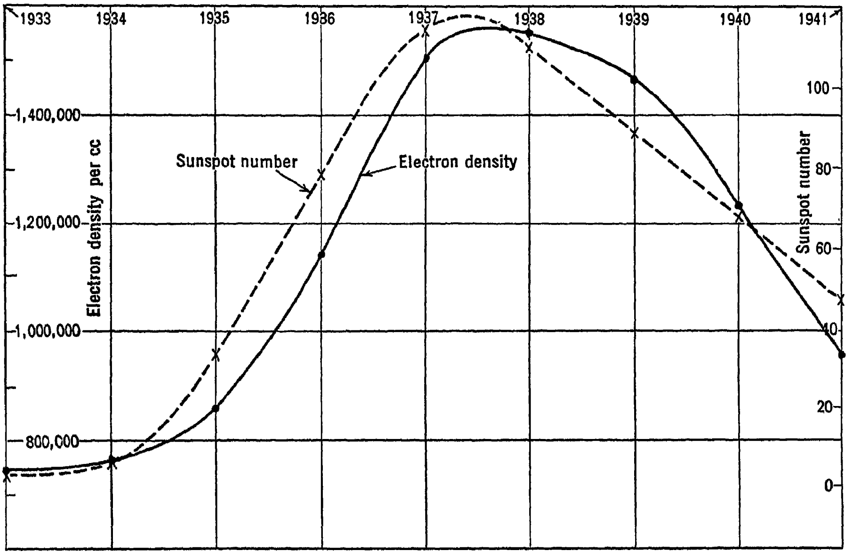


FIG. 3-10. Comparison of annual average sunspot number with annual average electron density of F_2 region measured at noon at Huancayo Magnetic Observatory, Peru.

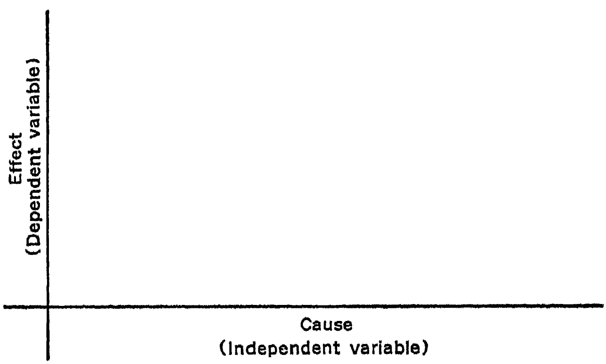


FIG. 3-11. Conventional assignment of variables to coordinate axes.

the other as an effect. This dependence of one variable upon another is interpreted graphically by plotting the *independent variable*, or cause,

along the x-axis, and the *dependent variable*, or effect, along the y-axis (Fig. 3–11). In certain circumstances it is difficult to dissociate cause and effect, and in these instances, unless some arbitrary standard has been adopted in practice, the choice of axes is immaterial. However, wherever dependence of one variable upon another is clear, the proper assignment of each to its appropriate axis is conventional.

Exercise 3–7

1. To determine the shelf life of a particular type of 45-volt dry battery a low resistance voltmeter was connected across the otherwise idle battery at intervals of about 60 days with the following results:

Number of days	0 (Battery new)	60	122	185	243	300	362
Voltmeter reading	48.5	47.9	47.7	47.5	47.4	47.2	47.0

Plot a graph showing the variation of voltage with time. *Extrapolate* (extend) the graph to estimate the voltage which might be expected after 2 years of idle storage. What can be said regarding the trustworthiness of this estimate? Why?

2. In the calibration of an oscilloscope for use as a voltmeter the position of the spot on the fluorescent screen was noted for various potentials applied to the deflecting plates. The position of the spot was measured by its distance from a fiducial mark on the glass. Following are the data obtained:

Applied potential in volts	0	10	20	30	40	50
Distance of spot from fiducial mark (centimeters)	1.6	3.4	5.1	6.9	8.6	10.4

Plot a graph showing how the deflection of the spot from the zero voltage position varies with the applied potential. Using this graph as a calibration chart of the instrument, find what voltage is indicated by a deflection from the zero voltage position of (a) 1.2 centimeters, (b) 6.8 centimeters.

3. The following corresponding output current and output potential values were obtained in the test of an unregulated power supply:

Current in milliamperes	10	20	45	75	150	205	230
Potential in volts	465	450	420	385	310	255	235

Plot a graph of output current versus output voltage for this power supply. By definition, regulation of a power supply is the ratio:

$$\frac{\text{no load voltage} - \text{full load voltage}}{\text{full load voltage}}.$$

What is the regulation of this power supply if it is rated at 200 milliamperes full load?

4. The following data represent the results of a laboratory test on two incandescent lamps, one a carbon filament lamp, the other a tungsten filament lamp:

Carbon Lamp	Current through lamp in milliamperes	50	100	150	200	250	300	335	360	400
	Potential across lamp in volts	17.0	28.5	39.6	51.0	61.9	73.7	81.5	87.5	97.0

Tungsten Lamp	Current through lamp in milliamperes	50	100	150	200	250	275	300	325	350	375	393
	Potential across lamp in volts	1.2	3.6	10.0	24.2	41.5	51.0	60.1	70.5	81.0	93.0	103.5

From these data calculate and tabulate the resistance in ohms (ratio of potential in volts to current in amperes) and the value of the square of the current for each case. Then plot on the same sheet the graphs of (a) resistance versus current for both lamps and (b) resistance versus current squared for both lamps. The square of the current at any time is an approximate indication of the relative temperature. Hence, the latter graphs show the nature of the variation of resistance with temperature for carbon and for tungsten.

From a study of the graphs of resistance versus current suggest an application of each type of lamp as a control device to regulate current or potential in a circuit.

3-10. Family of Curves. A group of related curves is known as a *family*. The characteristic which distinguishes one curve of a family from another is known as a *parameter*. Characteristic curves of vacuum tubes are often plotted in families. (See, for example, Fig. 25-1, Chapter 25.)

3–11. Requirements of a Coordinate System. A coordinate system is a scheme wherein each distinct set of numbers locates a particular point, and wherein each point is associated with a unique set of numbers. It is because of this one-to-one correspondence between number sets and points that either a graph (pictorial expression) or an equation (analytical expression) may be used to represent a relationship between variables. The coordinate system with which we have been working is known as a *rectangular coordinate system.* There are various other coordinate systems which provide one-to-one correspondence between points and number sets. However, because of its simplicity, the rectangular system is used almost to the exclusion of all other coordinate systems for general engineering work.

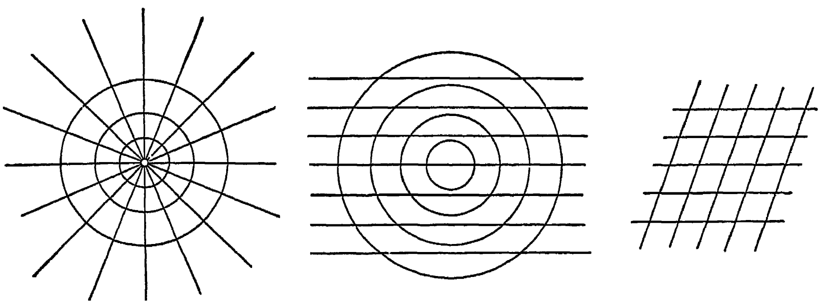


FIG. 3–12. Proposed designs for the basis of a coordinate system.

Exercise 3–8

Discuss the merits of each of the arrangements of Fig. 3–12 for use as a basis of establishing a one-to-one relationship between points and number-pairs.

3–12. Graphs for Reference. A number of graphs which are important in communications work are collected in Chapter 30. The student will find it advantageous to refer to these graphs from time to time in conjunction with related communications studies.

CHAPTER 4

PRACTICAL COMPUTATIONS

4–1. Relative Errors. A coal dealer is never required to guarantee the delivered weight of a ton of coal to within 1 or 2 pounds. Yet a butcher who operated with a similar margin of error on retail meat cuts would not be tolerated long in any community. Just what is it that makes for a greater tolerance in one case than in the other? The answer is that in both cases we accept about the same *relative*, or percentage, error. We realize that in the measurement of one physical quantity in terms of another, such as in the weight of an object in terms of the deflection of a scale pointer, it is impossible to attain any absolute degree of accuracy, and so we usually compromise on some reasonable *relative* error. An allowed 1 per cent error means a permissible deviation of 20 pounds in a ton of coal or of about $\frac{1}{2}$ ounce in 3 pounds of meat.

The relative accuracy obtainable with any particular instrument is usually best for measurements of quantities of large magnitudes. If it is possible for one to estimate tenths of a division in the scale reading of an electrical instrument, then the maximum likely reading error is one half of a tenth, or 0.05, of a scale division. In the measurement of a magnitude which corresponds to a pointer deflection from the zero mark of 10 scale divisions, an inaccuracy of 0.05 scale division means a relative error of $\frac{0.05}{10} = 0.5$ per cent. In the measurement of a magnitude which corresponds to a pointer deflection of 100 units, a reading error of 0.05 unit means a relative error of only $\frac{0.05}{10} = 0.05$ per cent. To obtain small relative reading errors it is good general practice to use only the upper two thirds of any electrical meter scale and to replace a meter by one of lower range, if practicable, in case a reading occurs in the lower third of the scale. This is particularly true in the case of certain alternating current instruments which are so calibrated that their scales are crowded at the lower end [Fig. 4–1(a)].

To simplify design problems practical engineers frequently make

deliberate approximations by which they knowingly introduce errors into computations if they can see that the consequent *relative* errors in their results are within the tolerable limits established by practice. The value 10 is often used in place of π^2, and sometimes the value 3 is used in place of π.

The expression

$$I = \frac{RE}{R^2 + X^2}$$

occurs in the study of a typical communications circuit. If the numerical values of R, X, and E are exactly 2, 100, and 50, respectively, then the evaluation of I yields

$$I = \frac{2 \cdot 50}{4 + 10{,}000} = 0.00996.$$

However, if we choose to neglect the term R^2, since it contributes only slightly to the sum $R^2 + X^2$, then we obtain

$$I = \frac{RE}{X^2} = \frac{2 \cdot 50}{10{,}000} = 0.01.$$

The value 0.01000 differs from the " true " value, 0.00996 by less than half a per cent (0.5 per cent). If, as is not unlikely, the value 0.00996 is in error by several times a half per cent, then the use of the approximate expression, $I = \dfrac{RE}{X^2}$, is justified whenever R is sufficiently small as compared with X.

Exercise 4-1

A. Estimate the percentage error in reading each of the meters shown in Fig. 4-1 corresponding to pointer deflections of the following magnitudes:

1. 45.3. 2. 22.7. 3. 8.4.

B. Compute the percentage error introduced on writing:

1. $\pi = 3$. 2. $\pi^2 = 10$.

C. Compute the percentage error introduced on neglecting the term R^2 in the evaluation of I from the expression $I = \dfrac{RE}{R^2 + X^2}$ in case E and X are numerically 50 and 100, respectively, and R is (a) 100, (b) 10, (c) 1.

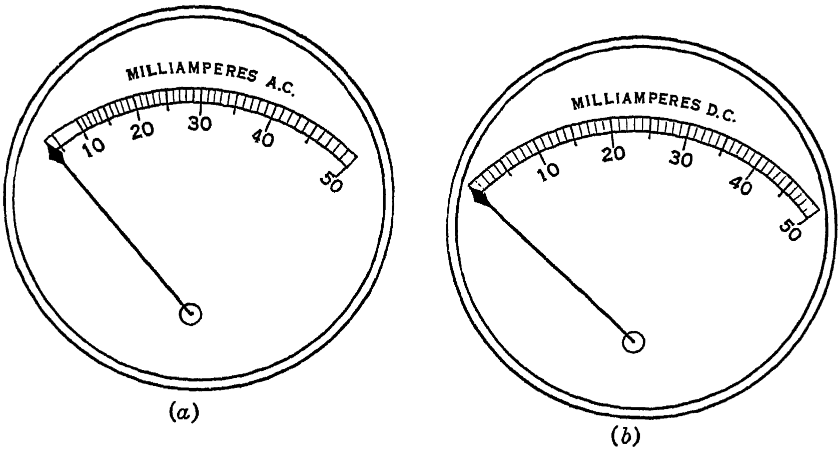


FIG. 4–1. Types of electrical instrument scales.

4–2. Interpretation of Laboratory Measurements. In electrical work many practical measurements are made to within about 1 per cent accuracy. In some cases, however, we cannot even estimate scale readings of the available instruments to better than 1 part in 50. And frequently the instruments themselves are inherently accurate only to within 5 per cent. It is well to bear in mind the limitations of the equipment when working with actual apparatus, and to adopt a sense of values accordingly.

Let us consider the following data, representing the result of resistance measurements on a circuit element for different values of current:

Current in milliamperes	10	20	30	40	50
Resistance in ohms	104	103	100	106	103

One way of plotting this information is shown in Fig. 4–2(a), where the scale of ordinates is chosen so as to begin at 100. This portrayal emphasizes the actual deviations of measured resistance from one point to another and tends to suggest that the resistance varies considerably with current. The second graph of the same data, in Fig. 4–2(b), with ordinates beginning at zero, shows these identical measurements in such

a manner as to indicate that the *relative* magnitudes of the deviations between points are really very small. The average measured resistance is

$$\frac{104 + 103 + 100 + 106 + 103}{5} = 103.2 \text{ ohms.}$$

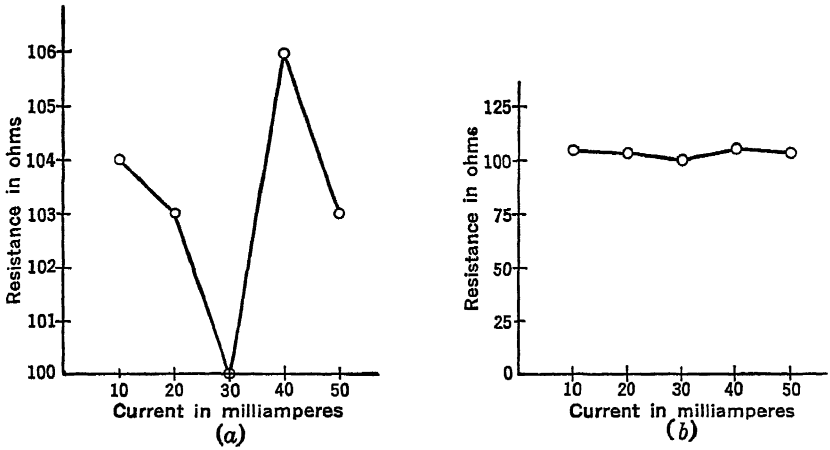


FIG. 4-2. Two graphs of the same experimental data,
(a) emphasizing actual magnitudes of deviations,
(b) showing relative magnitudes of deviations.

The maximum deviation from this value is $106 - 103.2 = 2.8$ ohms; the maximum percentage deviation is

$$\frac{2.8}{103.2} \cdot 100 = 2.7 \text{ per cent.}$$

If the test apparatus is inherently only 95 per cent accurate, then the fluctuations in the graphs of Fig. 4-2 do not permit of any direct interpretation with regard to the nature of the true resistance of the device being studied. Under these circumstances the experimenter concludes that within the limitations of the test apparatus employed the measured resistance is uniform.

4-3. Cumulative Errors. It may happen that the experimental determination of a certain quantity involves several measurements, each of which introduces some error, so that each measurement contributes to the net error in the result. Let us consider a situation, illustrated by

the circuit diagram of Fig. 4-3, in which it is desired to determine the voltage V_2 across a portion of a circuit as the difference of two other voltages V_3 and V_1. It may be necessary to resort to a measurement of this nature in the event that the voltmeter V_3 is permanently mounted in a chassis, and the resistor R_1, but not R_2, is easily accessible to the probes from the voltmeter V_1. If the voltage V_3 measures 120 volts to within 2 per cent error, the correct value of V_3 is between 120 − 2.4 volts and 120 + 2.4 volts. And if the voltage V_1 measures 50 volts to within 1 per cent error, the correct value of V_1 is between 50 − 0.5 volts and 50 + 0.5 volts. It is possible that, in the subtraction of the measured voltages V_3 and V_1 to yield the desired voltage V_2, the error in V_1 may partially compensate for the error in V_3, but in the worst case the errors are additive to give a maximum possible net error in V_2 of 2.4 + 0.5 = 2.9 volts. This means a net relative error in V_2 of

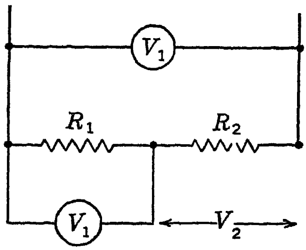


FIG. 4-3. Experimental determination of voltage across R_2.

$$\frac{2.9}{120 - 50} = 0.04 = 4 \text{ per cent.}$$

As a general statement, in an addition or subtraction of two experimentally measured values, the *actual* error in the result is at worst equal to the sum of the individual *actual* errors. In a product or quotient of experimentally measured values it may be demonstrated that the maximum *percentage* error in the result is approximately equal to the sum of the *percentage* errors in the individual measured values.

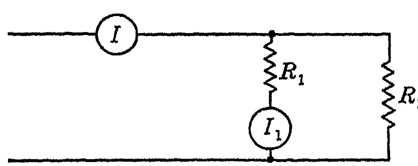


FIG. 4-4. Experimental determination of current through R_2.

Exercise 4-2

1. In the circuit of Fig. 4-4, the current in R_2 is to be determined as the difference of I, the line current, and I_1, the current through R_1. Observed values of I and I_1 are 5.31 amperes and 2.65 amperes, respectively, each plus or minus 2 per cent. Compute the current through R_2, and indicate the accuracy of the result.

2. The value of a resistance, R, may be experimentally determined by means of the relation (Ohm's Law)

$$R = \frac{E}{I},$$

where E and I are the observed potential across the resistance and current through the resistance, respectively. If the maximum percentage errors are 2 per cent in the measurement of E and 5 per cent in the measurement of I, how accurate is the resistance value obtained?

4-4. Significant Figures. An ordinary laboratory meter stick is calibrated in centimeters and in millimeters (tenths of centimeters). In the measurement of a length with the meter stick it is possible to estimate tenths of millimeters and to obtain, for example, such a value for a length measurement as 29.13 centimeters, wherein the last figure, 3, is estimated by the observer. Since the figure 3 is in doubt, any additional figures to the right of the 3 would, in general, be meaningless. A recorded value of 29.13 for a laboratory observation means 29.1 plus an estimated 0.03 and does not mean 29.13 exactly $(29.130000\cdots)$. This point is frequently overlooked when the results of different observations are combined.

Let us suppose that the aforementioned length measurement of 29.13 centimeters is one step in the experimental determination of a rectangular area, and that the width measurement of the area with the same meter stick is 7.42 centimeters, where the last figure, 2, is again an estimate. From the measurement it might be presumed that the area is in magnitude 216.1446 square centimeters, as would be obtained by direct multiplication:

$$\begin{array}{r} 29.13 \\ \underline{7.42} \\ 5826 \\ 11652 \\ \underline{20391} \\ 216.1446 \end{array}$$

On second thought, however, we see that this value is quite probably incorrect and is certainly misleading in its suggestion of four decimal-place accuracy. To assist us to remember that the figure 3 in the first measurement is doubtful, as is the figure 2 in the second measurement, let us write both figures with lines through them, thus: $\cancel{3}$ and $\cancel{2}$, and re-

examine our multiplication. Since both the 3 and the 4 are doubtful, any product involving either figure as a factor is doubtful. All doubtful figures are cross-lined in the multiplication below.

$$\begin{array}{r}
29.1\cancel{3} \\
\underline{7.4\cancel{2}} \\
\cancel{5}\cancel{8}\cancel{2}\cancel{6} \\
116\cancel{5}\cancel{2} \\
\underline{203\cancel{9}\cancel{1}} \\
21\cancel{6}.\cancel{1}\cancel{4}\cancel{4}\cancel{6}
\end{array}$$

Only the first two figures are certain! If our estimated last figure in the width measurement had been 5 instead of 2, we should have had for the formal product $29.13 \cdot 7.45 = 217.0185$. And if the estimated last figure in the width measurement had been 0 instead of 2, we should have had for the formal product $29.13 \cdot 7.40 = 215.5620$. Based on our measurements we can say only that the area is approximately 216 square centimeters, the last figure, 6, being uncertain.

With the exception of any zeros whose only purpose is to set the decimal point, all figures in a decimal expression up to and including the first doubtful figure are regarded as *significant figures*. Either the expression 156 or the expression 0.00156 is understood to imply three significant figures, namely, 1, 5, and 6. The expression 0.0015600 implies five significant figures, namely, 1, 5, 6, 0, 0. The last two zeros are meaningful — or significant — in that they show that the complete number is 0.0015600 as distinct from 0.0015601 or 0.0015599. An unqualified expression 78,000 may imply any one of several possibilities. The ambiguity may be removed by writing, for example, 78,000.00 in case seven significant figures are intended, or $780\cancel{0}\cancel{0}$ in case four significant figures are intended. In abstract mathematical calculations, that is, in calculations which are not directly related to physical measurements, the number 78,000 would probably imply exactly 78,000.000 · · ·.

It is only for the convenience of the study of significant figures in this chapter that we introduce the technique of drawing a line through a digit to indicate a doubtful figure wherever ambiguity might otherwise result. In Sec. 6-4 we shall observe a conventional method of denoting significant figures by the use of powers of 10.

Exercise 4–3

How many significant figures are there in each of the following numbers?

1. 10$\cancel{0}\cancel{0}$.
2. 101$\cancel{0}$.
3. 1001.
4. 100.0.
5. 0.35.
6. 0.035.
7. 0.0350.
8. 3.05.

4–5. Significant Figures in Multiplication and Division. In general, the number of significant figures in a product is equal to, or at best one greater than, the number of significant figures in that particular factor which of all the factors has the smallest number of significant figures. A similar rule holds for quotients, the number of significant figures being determined by whichever element, divisor, or dividend has the smaller number of significant figures. And there is usually no advantage of accuracy to be gained by employing more than these optimum numbers of significant figures for the factors in a product or for the dividend and the divisor in a quotient.

In dropping superfluous figures, if the first figure dropped is 5 or greater, the last retained figure should be increased by 1. Thus π, correct to six figures, is 3.14159; correct to five figures it is 3.1416.

In the multiplication of 29.13 by 7.42 in Sec. 4–4 the large number of cross-lined figures indicates that an appreciable amount of labor is wasted on unnecessary multiplication and division. A result of multiplying 29.13 by 7.42, which is equally as accurate as that found in Sec. 4–4, may be obtained by a simplified process. Having only three significant figures in 7.42, we use only three figures of 29.13 so that the labor is immediately reduced to that of multiplying 29.1 and 7.42. We first multiply 29.1 by 7:

$$\begin{array}{r} 291 \\ 7 \\ \hline 2037 \end{array}$$

Then we multiply 29 by 4:

$$\begin{array}{r} 29 \\ 4 \\ \hline 116 \end{array}$$

And finally we multiply 3 (closest whole number to 2.9) by 2:

$$\begin{array}{r} 3 \\ \underline{2} \\ 6 \end{array}$$

The complete process is tabulated as follows:

$$\begin{array}{r} 291 \\ \underline{742} \\ 2037 \\ 116 \\ \underline{6} \\ 2159 \end{array}$$

The position of the decimal point is determined by inspection, or by comparison with the result of a roughly equivalent simple operation, for example, by comparison with the product $30 \cdot 7 = 210$. The product $29.13 \cdot 7.42$ is, then, taken as 215.9.

Example. Multiply 316 by 247.

Multiply 316 by 2, and align the last digit of the product in the right column:

$$\begin{array}{l} 316 \\ \underline{2} \\ 632 \end{array}$$

Multiply 32 (which is 316 correct to two figures) by 4, and align the last digit of the product in the right column:

$$\begin{array}{l} 316 \\ \underline{24} \\ 632 \\ 128 \end{array}$$

Multiply 3 (which is 316 correct to one figure) and align the last digit of the product in the right column; add:

$$\begin{array}{r} 316 \\ \underline{247} \\ 632 \\ 128 \\ \underline{21} \\ 781 \end{array}$$

Set the decimal point by comparison with $300 \cdot 250 = 75{,}000$. The desired product is taken as 78,100.

It will be noted that even this method sometimes gives a result which implies more significant figures than are justified, but nevertheless it represents an improvement over the conventional multiplication scheme for dealing with experimental quantities.

Long division of two experimental values may be simplified as demonstrated in the following illustration of the division of 5765 by 4861. The first step is conventional:

$$\begin{array}{rr}
 & 1\\
4861 & \overline{)5765}\\
 & \underline{4861}\\
 & 904
\end{array}$$

For the next step, instead of performing the customary division of 4861 into 9040, wherein the last zero in 9040 is a meaningless digit, we perform the division of 486 into 904. And we continue in this manner at each successive step. The complete process is indicated below:

$$\begin{array}{rr}
 & 1.185\\
4861 & \overline{)5765}\\
 & 4861\\
486 & \overline{)\ 904}\\
 & 486\\
49 & \overline{)\ 418}\\
 & 392\\
5 & \overline{)\ \ 26}\\
 & \underline{25}\\
 & 1
\end{array}$$

The process self-indicates the extent to which the division may be carried within the accuracy obtainable from the initial quantities. An additional example of division follows.

Example. Divide 2321 by 57,382.

Here we have a choice of (*a*) retaining four figures in the divisor and introducing a zero in the dividend or (*b*) retaining only three figures in the divisor

and leaving the dividend as it is.

```
          .04045            .0404
   5738)23210          574)2321
         22952               2296
     57)   258            6)   25
           228                 24
      6)    30                  1
            30
             0
```

In the former case we retain more significant figures in the divisor, but in the latter case we avoid introducing a false figure in the dividend. The first case is to be favored for accuracy since here the *fifth* place of the dividend is affected (by introducing the false zero), whereas in the second case the *fourth* place of the divisor is affected (by dropping the fourth significant figure). In the first case we have four significant figures in both dividend and divisor; in the second case we have four significant figures in the dividend and three in the divisor.

Two zeros might be added to the dividend and all five figures could be retained in the divisor, but this is not practical in view of the fact that the dividend is good to only four figures.

Exercise 4-4

Perform each of the following operations taking cognizance of significant figures:

1. $28.1 \cdot 3.57$.
2. $0.00625 \cdot 1036$.
3. $12\cancel{0}\cancel{0} \cdot 17.3$.
4. $77.98 \div 0.0642$.
5. $0.0642 \div 77.98$.
6. $\dfrac{6205 \cdot 153}{0.0072}$.

4-6. Significant Figures in Addition and Subtraction. The number of significant figures in a sum or difference is apparent on inspection when the work is written in tabular form. Thus, the sum of the set of experimental values 28.07, 0.0013, and $3{,}8\cancel{1}\cancel{0}{,}\cancel{0}\cancel{0}\cancel{0}$ is seen to have three significant figures:

$$\begin{array}{r} 28.07 \\ 0.0013 \\ \underline{3{,}8\cancel{1}\cancel{0}{,}\cancel{0}\cancel{0}\cancel{0}} \end{array}$$

Since the first zero in the number 3,810,000 is a meaningless figure

(except in so far as it contributes toward setting the decimal point), any digit in the fourth or successive places in the sum is likewise a meaningless figure. Hence the "sum" is written as 3,810,000 or as $3{,}8\cancel{1}\cancel{0}{,}\cancel{0}\cancel{0}\cancel{0}$, and not as 3,810,028.0713.

The student should find that for the most part the remarks concerning accuracy and significant figures will be directly applicable in the laboratory. However, it must be borne in mind that rules concerning significant figures are at best generalizations, and that questions of accuracy require a certain amount of discretion on the part of the computer.

Exercise 4–5

Perform each of the following operations taking cognizance of significant figures:

1. 1200 + 17.3.
2. 0.094 − 15.40.

4–7. The Slide Rule. The slide rule is an almost indispensable aid to the computer. The basis of the construction of the slide rule and some detail concerning its manipulation will be presented in Chapter 17. For the present we shall find it worth while to learn the mechanical operations of simple multiplication and division in order that we may at once use the rule to this extent in our computations.

A typical commercial slide rule is shown in Fig. 4–5. The *slide* carrying the B, CI, and C scales is free to move in or out along the frame, or *body*, of the rule, which carries the A and D scales. The hairline scratch on the transparent *indicator* facilitates alignment of the markings from one scale to another.

4–8. Multiplication with the Slide Rule. Multiplication of two numbers is performed by setting the *left index* — the extreme left 1 mark — of the C scale opposite one factor on the D scale. The product is then found on the D scale opposite the second factor on the C scale. Only the significant figures in any quantity are regarded in operations with the slide rule; and the position of the decimal point in a result is determined by comparison of the result of a similar, simple computation.

Example 1. Multiply 2 by 3.4.

Set the left index of C to 2 on D. Opposite 34 on C find 68 on D. The position of the decimal point is determined by inspection. The result is 6.8.

Example 2. Multiply 103.6 by 0.397.

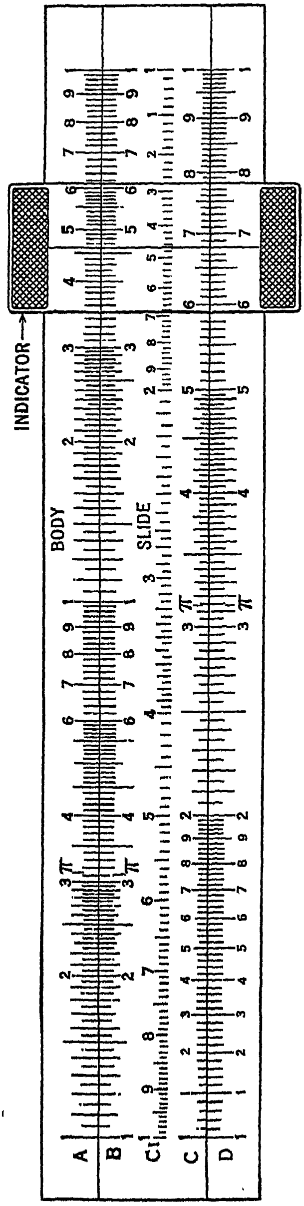


Fig. 4-5. Slide rule.

Set the left index of C to 1036 on D. The number 103 lies between 100, the left index, and 110. The number 110 on many rules is the first numbered mark to the right of the left index. The position of 1036, between 103 and 104, must be estimated. With the aid of the hairline find 411 on D opposite 397 on C. The position of 397 on C is estimated from the markings corresponding to 396 and 398. And the value 411 on D is estimated from the relative positions of the hairline and of the markings on D corresponding to 410 and 415. The location of the decimal point is determined from a consideration of the product $100 \cdot 0.4 = 40$. The desired product is, then, 41.1.

Exercise 4–6

Perform the following multiplications with the aid of a slide rule:

1. $15 \cdot 5.2$.
2. $0.015 \cdot 6.1$.
3. $21{,}000 \cdot 0.12$.
4. $1011 \cdot 453$.
5. $3.41 \cdot 2.07$.
6. $0.451 \cdot 142$.

4–9. Protrusion of the Slide in Multiplication. If we attempt to multiply certain numbers on a slide rule in the manner prescribed in Sec. 4–8, we find that a significant portion of the slide protrudes beyond the body of the rule. For example, to multiply 2 by 8 we might set the left index of C opposite 2 on D only to find that 8 on C lies beyond the right extremity of the D scale. In such a circumstance multiplication is performed by setting the *right index* — the extreme right 1* mark — of C opposite one factor on D. The product is found, as before, on D opposite the second factor on C.

Example. Multiply 6.6 by 4.1.

Set the right index of C to 66 on D. Opposite 41 on C find 271 on D. The location of the decimal point is here determined by inspection. The result is 27.1.

Exercise 4–7

Perform the following multiplications with the aid of a slide rule:

1. $7.13 \cdot 22.4$. (Use the right index of the C scale).
2. $401 \cdot 32.3$ (Use the right index of the C scale).
3. $401 \cdot 19.6$ (Use the left index of the C scale).
4. $0.266 \cdot 3.17$.
5. $8.23 \cdot 5.01$.
6. $17{,}100 \cdot 1203$.

* The extreme right mark on the C scale of some rules is a 10 instead of a 1.

4–10. Division with the Slide Rule. Division is accomplished with the slide rule in the inverse manner of multiplication. The divisor on C is set opposite the dividend on D, and the quotient is read on D opposite the on-scale index of C.

Example. Divide 35,000 by 12.6.

Set 126 on C opposite 350 on D. Opposite the left index of C find 278 on D. The position of the decimal point is ascertained by a comparison with the division 35,000 ÷ 10 = 3500. The desired quotient is, thus, 2780.

Exercise 4–8

A. Perform the following divisions with the aid of a slide rule:

1. 620 ÷ 159.
2. 3.75 ÷ 0.444.
3. 1.00 ÷ 8.35.
4. 5720 ÷ 0.0590.

B. The scales of the so-called 10-inch slide rule are not 10 inches, but 25 centimeters, in length. Using the slide rule, compute the actual scale length in inches. (1 inch = 2.54 centimeters.) Measure the scale length by comparison with an inch rule as a check.

4–11. Slide Rule Accuracy. Three digits may be read directly and a fourth digit estimated in a reading at the low end of a 10-inch slide rule scale. Two digits may be read directly and a third estimated at the high end of the scale. In any case the accuracy obtainable with the 10-inch slide rule is consistent with, or better than, the accuracy obtainable with most experimental apparatus, and in this respect the 10-inch slide rule is definitely a practical computing instrument.

4–12. Use of a Zero to Set Off a Decimal Point. It is common engineering practice to employ a zero preceding the decimal point in a decimal fraction. Thus, we write 0.1 instead of .1 and 0.0062 instead of .0062. This notation serves to dispel the likelihood of misunderstanding in that it indicates definitely in a decimal fraction that no units are intended preceding the decimal point. It is recommended that the student follow this practice in the recording of data in the laboratory and in the writing of engineering reports.

CHAPTER 5

ALGEBRAIC OPERATIONS

5-1. Symbols of Relationship. The following symbols are among those commonly employed to denote relationships between quantities:

Symbol	*Meaning*	*Example*
$\neq$ or $\ne$	Is not equal to	$2 + 2 \neq 5$
$\approx$ or $\cong$	Is approximately equal to	$\frac{1}{99} \approx \frac{1}{100}$
$>$	Is greater than	$3 > 2$
$<$	Is less than	$2 < 3$
$\geqq$	Is greater than or equal to	$1 \geqq \frac{1}{1 + x^2}$
$\leqq$	Is less than or equal to	$\frac{1}{1 + x^2} \leqq 1$
$\gg$	Is very much greater than	$100{,}000 \gg 1$
$\ll$	Is very much less than	$1 \ll 100{,}000$

Exercise 5-1

Join the following pairs of numbers with appropriate symbols other than the symbol of inequality:

1. 2; 5.
2. 5; 2.
3. -2; -5.
4. 40.122; 40.121.
5. 3; 40,000.
6. 10,000; 0.1.
7. 2; n, where n represents any negative integer.
8. 2; n, where n, represents any even positive integer.

5-2. Grouping Symbols. Grouping symbols are *parentheses* ( ), *brackets* [ ], *braces* { }, and the *horizontal bar* ——. The horizontal bar includes

the *fraction bar*, the top of the *radical sign* $\sqrt{}$, and the *vinculum*. The vinculum is a line drawn over a set of related quantities.

An illustration of the use of the vinculum is the notation $\overline{AB}$, which implies a line joining points A and B in contradistinction to the notation AB, which may designate a product, A times B. Illustrations of uses of other grouping symbols are $(a + b)^2$ to denote the product of the quantities $(a + b)$ and $(a + b)$; and $\dfrac{x + y}{p + q}$ to denote the quotient of the quantities $(x + y)$ and $(p + q)$. Grouping symbols are sometimes omitted when the context is clear without them.

Exercise 5-2

Evaluate:

1. $(2 + 1)^2 - \{(3 + 7) \div (6 - 1)\}$.
2. $[(5 - 1)^2 \div (7 - 9)]^2$.

5-3. Exponent Notation. Usually we speak of "the square of a number" instead of "the product of a number by itself," and we write 5^2 instead of $5 \cdot 5$. Similarly, we speak of "the cube of 5" instead of "the product of 5 times 5 times 5"; and we write 5^3 instead of $5 \cdot 5 \cdot 5$. These notations may be extended to apply to repeated multiplication in general. By definition:

$$
\begin{aligned}
a^1 &= a; \\
a^2 &= a \cdot a; \\
a^3 &= a \cdot a \cdot a; \\
&\vdots \\
a^n &= a \cdot a \cdot a \cdots a \ (n \text{ factors}).
\end{aligned}
$$

The superscript which designates the number of factors in a repeated product is called an *exponent*. Thus, n is the exponent in a^n; a^n is read: "the nth power of a" or "a to the nth power" or, simply, "a to the nth."

5-4. Sequence and Extent of Operations. An algebraic expression, as $a \cdot x \ + \ y \div c \ - \ d \cdot z$ is understood to imply

$$a \cdot x \quad + \quad y \div c \quad - \quad d \cdot z;$$

that is, first, a is multiplied by x, y is divided by c, and d is multiplied by z; then the product $d \cdot z$ is subtracted from the sum of the product $a \cdot x$ and the quotient $y \div c$. In general, whenever grouping symbols are not used in an algebraic expression it is understood that the points of

separation of the operations are provided by the $+$ and $-$ signs. This interpretation is emphasized on writing the above expression as

$$ax + \frac{y}{c} - dz,$$

wherein the multiplication dots are omitted, and the division sign is replaced by the fraction bar. Grouping signs may be used, if desired, further to stress the sequence of operations. Thus:

$$(ax) + \left(\frac{y}{c}\right) - (dz).$$

Any symbol, as, for example, an exponent which represents an operation on a quantity, is understood to affect only that quantity to which it is immediately adjacent. Thus, either $2 \cdot x^2$ or $2x^2$ means $2 \cdot x \cdot x$ and not $(2 \cdot x) \cdot (2 \cdot x)$; $\sqrt{3a} \cdot 6$ means $(\sqrt{3a}) \cdot 6$ and not $\sqrt{(3a \cdot 6)}$. To avoid possible misinterpretation an expression as $\sqrt{3a} \cdot 6$ might be written $6 \cdot \sqrt{3a}$.

5–5. Polynomials. An algebraic expression consisting of a single term, as $5ax^2$ or $3b$, is called a *monomial*. An algebraic expression consisting of two terms, as $5ax^2 + 3b$, is called a *binomial*; an expression of three terms, as $5ax^2 + 3b - 7$, is called a *trinomial*; and an expression of any number of terms, in general, is called a *polynomial*.

The *degree of a monomial* is the sum of the exponents of the *variables* occurring in it. The degree of $5x^2y$ is three; the degree of a^2bx is one. The *degree of a polynomial* is the degree of that term (monomial) which is of highest degree. The degree of $2x^3 + 5x$ is three; the degree of $x^2 - 7x + 10$ is two.

Exercise 5–3

A. Indicate the degree of each of the following monomials:

1. $2x$.
2. $5az^3$.
3. xyz^2.
4. 3^2u^5v.

B. Indicate the degree of each of the following polynomials:

1. $x^2 + 2x + 5$.
2. $xy + xyz$.
3. $2ay^6 + 4axy^3 + 5x$.
4. $x^3 + 3x^2y + 3xy^2 + y^3$.

5–6. Coefficient. In a monomial the product of any number of factors is called the *coefficient* of the product of the remaining factors. In $5ax^2$ the

coefficient of x^2 is $5a$. In $2b$ the coefficient of b is 2. In a term consisting of the product of constant factors and variable factors the word coefficient used alone implies the product of the constant factors. The coefficients in the expression $2ax + 7by$ are $2a$ and $7b$.

5-7. Commutative, Associative, and Distributive Laws. Many of us are wont to accept, without question of the necessity for proof, the validity of the following relations:

$$2 + 3 = 3 + 2^*;$$
$$2 + (3 + 4) = (2 + 3) + 4;$$
$$2 \cdot 3 = 3 \cdot 2;$$
$$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4);$$
$$2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4.$$

All of these relations may be demonstrated to be true not only as regards the numbers 2, 3, and 4, but for all numbers, positive and negative, rational and irrational.

In general terms, employing letters instead of numbers, the above relations may be stated as:

$$a + b = b + a \tag{5-1}$$
$$a + (b + c) = (a + b) + c \tag{5-2}$$
$$a \cdot b = b \cdot a \tag{5-3}$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{5-4}$$
$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{5-5}$$

Eq. (5-1) is the *cummutative law for addition;*
Eq. (5-2) is the *associative law for addition;*
Eq. (5-3) is the *commutative law for multiplication;*
Eq. (5-4) is the *associative law for multiplication;* and

* In writing $2 + 3 = 3 + 2$ we are applying the general rule that in an arithmetic addition the result obtained is independent of the order of the operations involved. That there are situations in which a result obtained is dependent upon the order of the operations involved is illustrated by the action of a loaded gun as a consequence of (1) cocking the piece and (2) pulling the trigger. A reversed sequence of operations definitely alters the result. Before we say that for any two numbers, a and b, the result of adding b to a is the same as that of adding a to b we should investigate the foundations of our number system and the meaning of addition to insure against encountering, for perhaps certain values of a and b, a result which does depend upon

Eq. (5-5) is the law which states that *multiplication is distributive with respect to addition.*

Inasmuch as the results are the same for both additions, $a + (b + c)$ and $(a + b) + c$, it is adequate to write, as we customarily do, simply $a + b + c$ to denote the sum of a, b, and c. Likewise, since the results are the same for both multiplications, $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$, it is adequate to write simply $a \cdot b \cdot c$ to denote the product of a, b, and c.

5-8. Addition of Polynomials. As a consequence of the commutative and associative laws of addition [Eqs. (5-1) and (5-2)] two polynomials may be added by adding their individual terms in any order. Thus, just as the arithmetic expression $(2 + 4) + (5 + 3)$ is equivalent to $(2 + 5) + (4 + 3)$, so the algebraic expression $(2x + y) + (x + 2y)$ is equivalent to $(2x + x) + (y + 2y)$ or $3x + 3y$.

For a systematic scheme of addition of polynomials we arrange each of the polynomials to be added so that like terms are in the same column and then add each column separately.

Example 1. Add $x + 3y - 5z$, $2x + z + 10$, and $-y + z$.

$$\begin{array}{rrrr}
x & + 3y & - 5z & \\
2x & & + z & + 10 \\
\hline
 & - y & + z & \\
\hline
3x & + 2y & - 3z & + 10
\end{array}$$

Example 2. Add $x^2 + 2x - 5$ and $-3x^2 - x + 7$.

$$\begin{array}{rrr}
x^2 & + 2x & - 5 \\
-3x^2 & - x & + 7 \\
\hline
-2x^2 & + x & + 2
\end{array}$$

Exercise 5-4

Perform the indicated additions:

1. $(x + y) + (2x + 3y)$.
2. $(5l - 2m + 1) + (-2l + 3m + 5)$.
3. $(3a + c) + (b - c) + (-a - b + c)$.
4. $(x^2 + 2x + 1) + (3x^2 + x + 5)$.
5. $(\frac{1}{2}a^2 + ab - b^2) + (3a^2 - 2ab + b^2)$.
6. $(2x^2 - x) + (x - 10) + (-x^2 + 5)$.

5-9. Removal of Grouping Signs. Since the sum of two polynomials $(a + b + c)$ and $(d + e + f)$ may be written as

$$a + b + c + d + e + f,$$

we may state as a general principle concerning removal of parentheses that parentheses which are preceded by a $+$ sign may be removed at will. To examine the situation for the case of parentheses which are preceded by a $-$ sign we consider the product of the monomial -1 and the polynomial $(a - b + c - d)$. The polynomial $(a - b + c - d)$ may be written as $[a + (-b) + c + (-d)]$. Then by Eq. (5-5)

$$(-1) \cdot (a - b + c - d) = (-1) \cdot [a + (-b) + c + (-d)] =$$
$$(-1) \cdot a + (-1) \cdot (-b) + (-1) \cdot c + (-1) \cdot (-d);$$

that is,

$$-(a - b + c - d) = -a + b - c + d.$$

We conclude that parentheses which are preceded by a $-$ sign may be removed provided the sign of each term within the parentheses is changed.

In removing grouping signs when an expression includes several sets of grouping signs contained one within another, it will be found advantageous to begin with the outermost one to avoid changing signs more than once. However, the final result is immaterial of the order in which the signs are removed.

Exercise 5-5

Remove grouping signs in the following expressions and simplify.

1. $(a + b) - (2a + b)$.
2. $-(a + b) - [a - (a - b)]$.
3. $2x + \{x + [5 - (3x + 1 - \overline{2x - 7}) + x] - 3\}$.

5-10. Subtraction of Polynomials. In solving Problem 1 of Exercise 5-5 we have performed the subtraction of $(2a + b)$ from $(a + b)$. We observe from this example that the subtraction of one polynomial from another may be accomplished by changing the sign of each term in the polynomial to be subtracted and adding it to the other polynomial.

Example. Perform the subtraction $(x + y - z) - (2x - y + 2z)$. Adding $x + y - z$ to $-2x + y - 2z$, we have

$$\begin{array}{r} x + y - z \\ -2x + y - 2z \\ \hline -x + 2y - 3z \end{array}$$

Hence $(x + y - z) - (2x - y + 2z) = -x + 2y - 3z$

Subtraction may be regarded as a special case of addition: the addition of two quantities of opposite signs. Thus,

$$2 - 3 = 2 + (-3),$$

$$a - b = a + (-b),$$

and now $(a + b + c) - (d + e + f) = (a + b + c) + [-(d + e + f)]$. The term *algebraic addition* implies either addition or subtraction. $+1$ and -1 *algebraically add* to equal zero.

Exercise 5-6

Perform the indicated subtractions:

1. $(2x + 3y) - (x + y)$.
2. $(x + y) - (2x + 3y)$.
3. $(p + q - 5) - (3q + p)$.
4. $(a^2 + 2ab + b^2) - (a^2 - b^2)$.

5-11. Multiplication of Monomials and Polynomials. As a consequence of the commutative and associative laws of multiplication [Eqs. (5-3) and (5-4)], the product of two monomials may be performed in an analogous manner to similar arithmetic operations. Thus, comparable to the arithmetic equation

$$(3 \cdot 2 \cdot 4^2) \cdot (6 \cdot 4 \cdot 2) = 2^2 \cdot 3 \cdot 4^3 \cdot 6$$

we have now the algebraic equation

$$(5xzy^2) \cdot (-2ayx) = -10ax^2y^3z.$$

We state as a general rule that multiplication of monomials may be accomplished by multiplying separately the coefficients and the individual letters and then expressing the products so obtained as factors in any convenient order.

Example. Multiply $3uv^2$ by $10u^2vw$.

$$\begin{array}{lrl} \text{Coefficients:} & 3 \cdot 10 & = 30. \\ \text{Individual letters:} & u \cdot u^2 & = u^3. \\ & v^2 \cdot v & = v^3. \\ & w & = w. \\ \text{Result:} & & 30u^3v^3w. \end{array}$$

The fact that multiplication is distributive with respect to addition [Eq. (5-5)] permits us to write the product of two polynomials $(a + b + c)$

and $(x + y)$ as

$$(a + b + c) \cdot (x + y) = (a + b + c)x + (a + b + c)y$$
$$= (ax + bx + cx) + (ay + by + cy).$$

We state as a general rule that two polynomials may be multiplied by multiplying each term of one polynomial by each term of the other and adding the individual products so obtained.

Example. Multiply $(x - 2y + z)$ by $(3x - y - z)$.

In multiplying term by term let us multiply first the first term of $(x - 2y + z)$ by each term of $(3x - y - z)$, then the second term of $(x - 2y + z)$ by each term of $(3x - y - z)$, and then the third term of $(x - 2y + z)$ by each term of $(3x - y - z)$:

$$\begin{aligned} x \cdot (3x - y - z) &= 3x^2 - xy - xz. \\ -2y \cdot (3x - y - z) &= -6xy + 2y^2 + 2yz. \\ z\,(3x - y - z) &= 3xz - yz - z^2. \end{aligned}$$

The complete product is the sum of the partial products:

$$\begin{array}{cccccc} 3x^2 & - \;xy & - \;xz & & & \\ & -6xy & & + \;2y^2 & + \;2yz & \\ & & 3xz & & - \;yz & -z^2 \\ \hline 3x^2 & -7xy & + \;2xz & + \;2y^2 & + \;yz & - \;z^2 \end{array}$$

Exercise 5-7

Perform the indicated multiplications:

1. $2xr \cdot 3ax$.
2. $3x(1 - x)$.
3. $2m(m^2 + 5m - 7)$.
4. $(c + d)(r + s)$.
5. $(x + 1)(x - 2)$.
6. $(2x + 3)(x + 4)$.
7. $(5ax + 1)(3ax - 2)$.
8. $(4 + n)(n - 2)$.
9. $(t^2 + 2t + 1)(t + 3)$.
10. $(x + y + b)(x + y - b)$.

5-12. Frequently Occurring Products. The following products occur frequently in engineering practice:

$$(x + y)^2 = x^2 + 2xy + y^2. \tag{5-6}$$

$$(x - y)^2 = x^2 - 2xy + y^2. \tag{5-7}$$

$$(x + y)(x - y) = x^2 - y^2. \tag{5-8}$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab. \tag{5-9}$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3. \tag{5-10}$$

Exercise 5–8

Perform the multiplications of Eqs. (5–6) through (5–11).

5–13. Multiplication of Fractions. As in arithmetic the product of two fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of the given fractions. Thus,

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

We might devise a general proof for this statement as follows.

$$\frac{a}{b}\cdot\frac{c}{d}\cdot bd=\left(\frac{a}{b}\cdot b\right)\cdot\left(\frac{c}{d}\cdot d\right).$$

Now, regarding division as the inverse of multiplication, we have

$$\frac{a}{b}\cdot b=a, \quad \text{and} \quad \frac{c}{d}\cdot d=c;$$

also

$$\frac{ac}{bd}\cdot bd=ac.$$

Hence,

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot bd=ac=\left(\frac{ac}{bd}\right)\cdot bd,$$

from which we conclude that

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

Example. Multiply $\frac{2m}{xy}$ by $\frac{-3m^2}{x^2z}$.

$$\frac{2m}{xy}\cdot\frac{-3m^2}{x^2z}=\frac{-6m^3}{x^3yz}.$$

Exercise 5–9

Perform the indicated multiplications:

1. $\frac{5ab}{pq}\cdot\frac{rs}{2}$.
2. $3x^2z\cdot\frac{5z}{11}$.
3. $\frac{-7xy}{z}\cdot\frac{3xy^2}{z^2}$.
4. $\frac{3x^2}{a}\cdot\frac{2xy}{ab}\cdot\frac{4xy^3}{b^2}$.

5-14. Division of Monomials. The observations of Sec. 5-13 enable us to divide two monomials as illustrated by the following:

$$\frac{10abxy^2z^5}{2ax^2z} = \frac{10}{2}\cdot\frac{a}{a}\cdot\frac{by^2}{1}\cdot\frac{x}{x^2}\cdot\frac{z^5}{z},$$

$$= 5\cdot 1\cdot by^2\cdot\frac{1}{x}\cdot z^4,$$

$$= \frac{5by^2z^4}{x}.$$

In the foregoing all factors which are common to both numerator and denominator of the original fraction are missing from the result. A fraction is said to be in its *lowest terms* when all factors which are common to both numerator and denominator have been removed.

Exercise 5-10

A. Perform the indicated divisions:

1. $\dfrac{15a^2b}{5ab}$. 2. $\dfrac{-28a^2x^2z}{4a^3z}$.

B. Multiply and reduce to lowest terms:

1. $\dfrac{x}{z}\cdot\dfrac{z}{x}$. 2. $\dfrac{2a}{3b}\cdot\dfrac{9b^2}{4a^2}$.

C. Show that the value of any fraction is unchanged if both numerator and denominator of the fraction are multiplied by the same number (zero excluded).

5-15. Division of Fractions. As in arithmetic the division of two fractions may be accomplished by inverting the divisor and multiplying. We might prove this in general by noting, first, that

$$\left(\frac{\frac{a}{b}}{\frac{c}{d}}\right)\cdot\frac{c}{d} = \frac{a}{b}$$

and, secondly, that

$$\frac{a}{b}\cdot\frac{d}{c}\cdot\frac{c}{d} = \frac{a}{b}\cdot\frac{dc}{cd} = \frac{a}{b}.$$

Both $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ and $\dfrac{a}{b}\cdot\dfrac{d}{c}$ when multiplied by $\dfrac{c}{d}$ yield $\dfrac{a}{b}$, from which we conclude that

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\cdot\frac{d}{c}.$$

Exercise 5-11

Perform the indicated divisions:

1. $\dfrac{3a^2b}{5cd} \div \dfrac{9ab^2}{20c^2}$.

2. $\dfrac{24a^2}{33b} \div \dfrac{12a}{55b^2}$.

5-16. Division of Polynomials. The division of two polynomials may be accomplished by proceeding in a manner analogous to that employed in Sec. 5-14 for the division of two monomials (see Sec. 5-19 on Factoring), or it may be accomplished by regarding division as a series of successive subtractions (as in common arithmetic long division). The latter method is illustrated by the following examples.

Example 1. Divide $2x^2 - x - 10$ by $x + 2$.

$$\begin{array}{r|l}
 & \quad 2x - 5 \\
\hline
x + 2 & 2x^2 - x - 10 \\
 & \underline{2x^2 + 4x} \\
 & - 5x - 10 \\
 & \underline{- 5x - 10}
\end{array}$$

Here $x + 2$ is first multiplied by $2x$ to yield $2x^2 + 4x$; $2x^2 + 4x$ when subtracted from $2x^2 - x - 10$ leaves $-5x - 10$. And $x + 2$ times -5 exactly equals $-5x - 10$ leaving a zero remainder on subtraction.

Example 2. Divide $3x^3 + x^2 + 1$ by $2x + 1$.

$$
\begin{array}{r|l}
 & \frac{3}{2}x^2 - \frac{x}{4} + \frac{1}{8} \\
\cline{2-2}
2x + 1 & 3\,x^3 + x^2 + 1 \\
 & 3x^3 + \frac{3}{2}x^2 \\
\cline{2-2}
 & \qquad - \frac{x^2}{2} + 1 \\
 & \qquad - \frac{x^2}{2} - \frac{x}{4} \\
\cline{2-2}
 & \qquad\qquad\qquad \frac{x}{4} + 1 \\
 & \qquad\qquad\qquad \frac{x}{4} + \frac{1}{8} \\
\cline{2-2}
 & \qquad\qquad\qquad\quad\ \frac{7}{8}
\end{array}
$$

Thus,

$$\frac{3x^2 + x^2 + 1}{2x + 1} = \frac{3}{2}x^2 - \frac{x}{4} + \frac{1}{8} + \frac{7/8}{2x + 1}.$$

It is convenient that the terms of both the dividend and divisor be arranged as shown in the above examples in descending powers of a common letter.

Exercise 5–12

Perform the indicated divisions:

1. $2x^2 + x - 3 \div 2x + 3.$

2. $2m^2 + 6m^3 - m - 1 \div 3m - 2.$

3. $x^2 - 5 \div x + 2.$

4. $2u^3 - 2u^2 + u + 5 \div u - 1.$

5–17. Addition of Fractions. Two fractions which have a common denominator may be added as in arithmetic by adding numerators and retaining the common denominator. Thus,

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

The process may be regarded as equivalent to the following:

$$a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = (a + b) \cdot \frac{1}{c} = \frac{a + b}{c}.$$

Exercise 5–13

Perform the indicated additions:

1. $\dfrac{C_1}{V} + \dfrac{C_2}{V}$.
2. $\dfrac{3}{ab^2} - \dfrac{2}{ab^2} + \dfrac{5}{ab^2}$.
3. $\dfrac{4s}{r} + \dfrac{x}{r} - \dfrac{3s}{r}$.

5–18. Common Denominator. The fractions $\dfrac{a}{l}$, $\dfrac{b}{m}$, and $\dfrac{c}{n}$ may be expressed as

$$\frac{a}{l}\cdot\frac{mn}{mn} = \frac{amn}{lmn},$$

$$\frac{b}{m}\cdot\frac{ln}{ln} = \frac{bln}{lmn},$$

$$\frac{c}{n}\cdot\frac{lm}{lm} = \frac{clm}{lmn}.$$

As a general rule, to express fractions in terms of a common denominator, we:

1. Take the product of the given denominators as a common denominator.
2. Divide the common denominator by the denominator of the first given fraction.
3. Multiply the quotient obtained in (2) by the numerator of the first given fraction to obtain the numerator of the first desired fraction.
4. Construct the first desired fraction from the numerator found in (3) and the common denominator found in (1).
5. Repeat (2), (3), and (4) with each of the other given fractions.

Example. Express the following fractions in terms of a common denominator:

$$\frac{a}{3xy}, \frac{b}{6xz}, \frac{c}{2yz}.$$

1. We take for a common denominator $(3xy)\cdot(6xz)\cdot(2yz)$.
2. Dividing this common denominator by the denominator of the first given fraction, we obtain

$$\frac{(3xy)\cdot(6xz)\cdot(2yz)}{3xy} = (6xz)\cdot(2yz).$$

3. Multiplying the quotient obtained in (2) by the numerator of the first given fraction, we have $a\cdot(6xz)\cdot(2yz)$ for the numerator of the first desired fraction.

4. The first fraction becomes

$$\frac{a}{3xy} = \frac{a \cdot (6xz) \cdot (2yz)}{(3xy) \cdot (6xz) \cdot (2yz)}.$$

5. For the second fraction

$$\frac{(3xy) \cdot (6xz) \cdot (2yz)}{6xz} = (3xy) \cdot (2yz),$$

so that the second fraction becomes

$$\frac{b}{6xz} = \frac{b \cdot (3xy) \cdot (2yz)}{(3xy) \cdot (6xz) \cdot (2yz)}.$$

For the third fraction

$$\frac{(3xy) \cdot (6xz) \cdot (2yz)}{2yz} = (3xy) \cdot (2yz),$$

so that the third fraction becomes

$$\frac{c}{2yz} = \frac{c \cdot (3xy) \cdot (2yz)}{(3xy) \cdot (6xz) \cdot (2yz)}.$$

Exercise 5-14

Perform the indicated additions. Give each result in its lowest terms.

1. $\dfrac{x}{2} + \dfrac{y}{6} + \dfrac{z}{3}$.
2. $\dfrac{3x}{4} + \dfrac{2x}{3} - \dfrac{5x}{12}$.
3. $\dfrac{1}{R_1} + \dfrac{1}{R_2}$.
4. $\dfrac{10}{3a} + \dfrac{3}{4a} - \dfrac{1}{2a}$.
5. $\dfrac{2w}{uv} - \dfrac{u}{4vw} + \dfrac{v}{uw}$.

5-19. Factoring. Multiplication of a by $(x + y + z)$ yields $ax + ay + az$. Hence, we have

$$ax + ay + az = a(x + y + z). \tag{5-12}$$

Now, if we are given the expression $3r + 3s + 3t$, we may write at once by analogy with Eq. (5-12),

$$3r + 3s + 3t = 3(r + s + t).$$

Further, since multiplication of $(a + b)$ by $(x + y)$ yields

$$a(x + y) + b(x + y) = ax + ay + bx + by,$$

we have — working back from the product to the factors —

$$ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y). \quad (5\text{–}13)$$

And if we are given the expression $2p + 2q + kp + kq$, we may write at once by analogy with Eq. (5–13),

$$2p + 2q + kp + kq = 2(p + q) + k(p + q) = (2 + k)(p + q).$$

The determination of those factors whose product equals a given expression is known as *factoring*. Inasmuch as factoring is the reverse of multiplication, we find it useful in factoring to have in mind several typical product forms, such as those of Eqs. (5–6) through (5–11).

Example 1. Factor $4x - 6$.
We observe that the factor 2 is common to both terms. Hence,

$$4x - 6 = 2(2x - 3).$$

Example 2. Factor $x^2 - x - 6$.
The expression $x^2 - x - 6$ is of the form $x^2 + (a + b)x + ab$, which we found in Eq. (5–9) arises on the multiplication of $(x + a)$ and $(x + b)$. By comparing the factorable expression, $x^2 + (a + b)x + ab$, with the given expression, $x^2 - x - 6$, we see that our problem of factoring $x^2 - x - 6$ amounts to finding two quantities, a and b, which are such that

$$a + b = -1,$$

and

$$ab = -6.$$

A general method of solving the pair of equations $a + b = -1$ and $ab = -6$ will be given in Chapter 8. Here we find simply by inspection that $a = -3$ and $b = 2$. This means that $x^2 - x - 6$ may be written as

$$x^2 + (-3 + 2)x + [(-3) \cdot 2],$$

so that its factors by Eq. (5–9) are $x - 3$ and $x + 2$.
Multiplying, to check, we obtain

$$(x - 3)(x + 2) = x^2 + 2x - 3x - 6 = x^2 - x - 6.$$

Example 3. Factor $ac + bc + ad + bd$. Grouping the first two terms and the last two terms we have

$$(ac + bc) + (ad + bd).$$

This expression, on factoring c from $(ac + bc)$ and on factoring d from $(ad + bd)$, becomes

$$c(a + b) + d(a + b).$$

The quantity $(a + b)$ occurs in both terms and may be factored out giving

$$(c + d)(a + b).$$

Exercise 5–15

A. Factor the following expressions, verifying the result in each case by multiplying the factors to obtain the original expression:

1. $2x + 6y$.
2. $x^3y^2 - x^2y^3$.
3. $x(x - y) - b(y - x)$.
4. $2mx - nx - 2my + ny$.
5. $x^3 + 3x + 2$.
6. $t^2 + 4t + 4$.
7. $p^2 - q^2$.
8. $4x^2 - 9$.
9. $6 - 5x + x^2$.
10. $10 + 7a + a^2$.

B. Show that if $\frac{a}{b} = \frac{c}{d}$, then

$$\frac{a}{b} = \frac{c}{d} = \frac{a + c}{b + d}.$$

Suggestion: Write $\frac{a}{b} = \frac{c}{d} = r$, so that $a = br$ and $c = dr$.

5–20. Lowest Common Denominator. It is usually convenient for the addition of several fractions to express each fraction in terms of the *lowest common denominator.* For a group of fractions, each of whose denominator is a polynomial, the lowest common denominator is that polynomial of the lowest degree which contains each of the denominators as a factor. The fractions $\frac{1}{x}$, $\frac{1}{x^2}$, and $\frac{1}{1 + x}$ may be expressed in terms of a common denominator $x \cdot x^2(1 + x)$, or in terms of a common denominator $x^2(1 + x)$. The latter expression is the polynomial of lowest degree which contains each denominator as a factor and, hence, it is the lowest common denominator.

To express fractions in terms of their lowest common denominator we follow the steps (2) through (5) prescribed for any common denominator in Sec. 5–18.

Example. Express the following fractions in terms of their lowest common denominator:

$$\frac{a}{3xy}, \frac{b}{6xz}, \frac{c}{2yz}.$$

(These are the same fractions which were considered in the example of Sec. 5–18.)

1. The lowest common denominator is $6xyz$. This expression is of the first degree in each of the variables x, y, and z; and yet it contains each denominator as a factor.
2. Dividing the lowest common denominator by the denominator of the first given fraction, we obtain

$$\frac{6xyz}{3xy} = 2z.$$

3. Multiplying the quotient obtained in (2) by the numerator of the first fraction, we have $2az$ for the numerator of the first desired fraction.
4. The first fraction becomes

$$\frac{a}{3xy} = \frac{2az}{6xyz}.$$

5. For the second fraction

$$\frac{6xyz}{6xz} = y,$$

so that the second fraction becomes

$$\frac{b}{6xz} = \frac{by}{6xyz}.$$

For the third fraction

$$\frac{6xyz}{2yz} = 3x,$$

so that the third fraction becomes

$$\frac{c}{2yz} = \frac{3cx}{6xyz}.$$

Exercise 5-16

Perform the indicated additions, first expressing each fraction in terms of the lowest common denominator:

1. $\frac{5}{x} + \frac{2}{x^2} - \frac{3}{x^3y}$.
2. $\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}$.
3. $\frac{4y - x}{xy} + \frac{x - 4z}{xz} - \frac{1}{y}$.
4. $\frac{z}{p - q} - \frac{2q}{p^2 - pq}$.
5. $\frac{1}{1 - x} + \frac{1}{x} + \frac{x}{x - 1}$.

5-21. Transposition. The addition of 3 to both sides of the equation $x - 3 = 5$ gives

$$x - 3 + 3 = 5 + 3,$$

or

$$x = 5 + 3.$$

The net result is the same as though the 3 had been simply transferred from one side of the given equation to the other with its sign changed. This operation, referred to as *transposition*, expedites the solution of certain problems.

Example. Solve the equation $3(x - 5) = x - 2(x + 7)$.
Performing the indicated multiplications, we obtain

$$3x - 15 = x - 2x - 14.$$

Transposing, we get $\quad 3x - x + 2x = 15 - 14,$

from which it follows that $\quad 4x = 1,$

and $\quad x = \frac{1}{4}.$

Exercise 5-17

Solve the following equations:

1. $x - 8 = 10 - 2x$.
2. $9(7x - 6) = 5 - 8(3 - 7x)$.
3. $(x + 5)(x - 1) = (x + 3)(x - 2) + 13$.
4. $(x + 3)^2 - (x + 2)^2 = 1$.

5-22. Cross Multiplication. The multiplication by $\dfrac{2}{5}$ of both sides of the equation $\dfrac{5x}{2} = \dfrac{10}{3}$ gives

$$\frac{5x}{2} \cdot \frac{2}{5} = \frac{10}{3} \cdot \frac{2}{5},$$

or

$$x = \frac{10 \cdot 2}{3 \cdot 5}.$$

The net result here is the same as we might have obtained on simply transferring the 5 from the numerator of the left side to the denominator of the right side and the 2 from the denominator of the left side to the numerator of the right side. Although the logical process is one of

multiplication of both sides of an equation by the same number, the actual manipulations involve merely a transfer of factors from one side of the equation to the other. This formal operation is referred to as *cross multiplication.*

Example. Solve the equation $2 = \frac{3}{x}\cdot$

Cross multiplying, we obtain

$$x = \frac{3}{2}\cdot$$

Exercise 5-18

Solve the following equations for x:

1. $\frac{x}{6} = -7.$

2. $-\frac{8}{x} = \frac{3}{4}\cdot$

3. $\frac{2}{5}(2x + 1) = 3.$

4. $\frac{1}{x+2} = 2.$

5-23. Functional Notation. It is often convenient to describe a relation between two variables, x and y, by saying that y is a *function of* x or, in writing, $y = f(x)$ (read: "y equals f of x"). The equation $y = f(x)$ is a way of expressing a dependence of y upon x, and the letter f serves as a symbol of the nature of the dependence. We might choose $f(x)$ to represent $x + 2$, and $F(x)$ might be chosen to represent $\frac{1}{1+x}\cdot$ The symbol preceding the parentheses, f or F in the above cases, denotes the nature of the expression, and the letter within the parentheses denotes the variable or the quantity which is operated on. Thus, if we agree to represent $x + 2$ by $f(x)$ and $\frac{1}{1+x}$ by $F(x)$, then we have: $f(z) = z + 2$, and $F(z) = \frac{1}{1+z}$; $f(0) = 2$, and $F(0) = 1$; $f(3) = 5$, and $F(3) = \frac{1}{4}\cdot$

The functional notation may be extended to include expressions involving several variables. For example $\phi(x,y)$ may designate $x^2 + 3y$. The symbol ϕ here denotes a particular expression in x and y, that is, a function of x and y. The symbol $\phi(a,b)$ denotes the function $\phi(x,y)$ with a replacing x everywhere it appears and with b replacing y. Thus, if $\phi(x,y) = x^2 + 3y$, then $\phi(a,b) = a^2 + 3b$, and $\phi(-1,2) = 1 + 3\cdot 2 = 7$.

Exercise 5–19

A. If $f(x) = x + 2x + 3$, express:

1. $f(a)$. **2.** $f(u)$. **3.** $f(x + h)$.

B. If $\phi(x) = \dfrac{x^2}{3}$ evaluate:

1. $\phi(3)$. **2.** $\phi(-1)$. **3.** $\phi(1.5)$.

C. If $F_1(u,v,w) = u + 2uv - 3w$, evaluate:

1. $F_1(1,1,1)$. **2.** $F_1(1,-1,0)$. **3.** $F_1\left(0,5,\dfrac{1}{3}\right)$.

D. If $f(x,y) = x^2 + 2y + y^2$, for what values of x and y does:

1. $f(x,y) = f(-x,y)$?
2. $f(x,y) = f(x,-y)$?
3. $f(-x,y) = f(x,-y)$?

CHAPTER 6

EXPONENTS

6–1. Laws of Exponents. In this chapter the letters a and b will indicate positive quantities.

Computations with exponents are based on the following three laws which are known as the *Laws of Exponents:*

If m and n are any two positive integers, then

$$a^m \cdot a^n = a^{m+n}; \tag{6–1}$$

$$(a^m)^n = a^{mn}; \tag{6–2}$$

and

$$(ab)^m = a^m b^m. \tag{6–3}$$

To prove Eq. (6–1) we observe that

$$\begin{aligned} a^m \cdot a^n &= (a \cdot a \cdot a \cdots \text{to } m \text{ factors})\ (a \cdot a \cdot a \cdots \text{to } n \text{ factors}) \\ &= a \cdot a \cdot a \cdots \text{to } m + n \text{ factors} \\ &= a^{m+n}. \end{aligned}$$

The proofs of Eqs. (6–2) and (6–3) are left as an exercise (Exercise 6–1 below).

Exercise 6–1

1. Prove Eq. (6–2).
2. Prove Eq. (6–3).

6–2. Roots. The *nth root of a*, denoted by $\sqrt[n]{a}$, is a number whose nth power is a. An expression of the form $\sqrt[n]{a}$ is known as a *radical*; n is called the *index* and a the *radicand* of the radical. If an index is not specified, it is understood to be 2. The *principal nth root of a* is that *positive* number whose nth power is a. The second, or square, roots of 25 are 5 and -5. The principal second root of 25 is 5. In communications literature $\sqrt[n]{a}$ usually denotes only the principal nth root of a. Where it is desired to express both square roots of a, one writes $\sqrt{a}$ and

$-\sqrt{a}$. Where it is desired to express *either* of the square roots of a, one writes $\pm\sqrt{a}$, the symbol $\pm$ meaning "plus or minus."

Computations with radicals are based on the following formulas:

$$\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}; \tag{6-4}$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}; \tag{6-5}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}; \tag{6-6}$$

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}; \tag{6-7}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}. \tag{6-8}$$

Eqs. (6-4) through (6-8) follow from the Laws of Exponents and from the fact that two positive numbers are equal if any like powers of these numbers are equal.

To prove Eq. (6-4) we observe that, by definition,

$$(\sqrt[n]{a^m})^n = a^m,$$

and

$$(\sqrt[np]{a^{mp}})^{np} = a^{mp}.$$

From the first of these two equations, together with Eq. (6-2), it follows that

$$[(\sqrt[n]{a^m})^n]^p = (\sqrt[n]{a^m})^{np} = a^{mp}.$$

Thus,

$$(\sqrt[n]{a^m})^{np} = (\sqrt[np]{a^{mp}})^{np},$$

or

$$\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}.$$

To prove Eq. (6-5) we observe that

$$(\sqrt[n]{ab})^n = ab;$$

further, that

$$(\sqrt[n]{a})^n = a$$

and

$$(\sqrt[n]{b})^n = b;$$

so that

$$(\sqrt[n]{a})^n \cdot (\sqrt[n]{b})^n = ab.$$

The proofs of Eqs. (6–6), (6–7), and (6–8) are left as an exercise (Exercise 6–2 below).

Exercise 6–2

1. Prove Eq. (6–6).
2. Prove Eq. (6–7).
3. Prove Eq. (6–8).

6–3. Extension of the Laws of Exponents. Computations are frequently simplified if we write $a^{\frac{p}{q}}$ instead of $\sqrt[q]{a^p}$, and a^{-r} instead of $\frac{1}{a^r}$; because with these changes all of the equations (6–1) through (6–8) may be shown to be included in the three equations (6–1) through (6–3), which we call the Laws of Exponents.

To demonstrate that the Laws of Exponents may thus be extended so as to include all of the equations (6–1) through (6–8) we are obliged to consider all the possible cases of each law individually. For instance, we must examine four possible cases of the equation

$$a^m \cdot a^n = a^{m+n},$$

where m and n are now not simply integral and positive, but rational and of either sign. We must consider situations wherein: (1) both exponents are positive, (2) both exponents are negative, (3) the exponents are of opposite signs with the positive one numerically greater than the negative one, and (4) the exponents are of opposite signs with the negative one numerically greater than the positive one. If we put $m = \frac{p}{q}$ and $n = \frac{r}{s}$, where r and s are positive integers, the proof for case (1) is as follows:

$$\begin{aligned} a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} &= \sqrt[q]{a^p} \cdot \sqrt[s]{a^r} = \sqrt[qs]{a^{ps}} \cdot \sqrt[qs]{a^{qr}} \\ &= \sqrt[qs]{a^{ps} \cdot a^{qr}} = \sqrt[qs]{a^{ps+qr}} \\ &= a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}. \end{aligned}$$

Remaining proofs of the generality of the Laws of Exponents are left as an exercise (Exercise 6–3 below).

Exercise 6-3

A.

1. Complete the proof of Sec. 6-3 to show that Eq. (6-1) holds for all rational exponents.
2. Prove that Eq (6-2) holds for all rational exponents.
3. Prove that Eq. (6-3) holds for all rational exponents.
4. Show that to be consistent with Eq. (6-1) $a^0 = 1$ if $a \neq 0$. 0^0 is undefined. Why?

B. Simplify the following expressions by the use of fractional exponents and/or negative exponents.

1. $\sqrt{a \cdot \sqrt{a}}$.
2. $\dfrac{\sqrt[3]{ab} \cdot \sqrt{b^5}}{a^2\sqrt{b}}$.
3. $\dfrac{(\sqrt{R^2 + X^2})^3}{\sqrt{R^2 + X^2}}$.

C. Evaluate mentally:

1. $\dfrac{6^5}{6^3}$.
2. $\dfrac{7^3}{7^5}$.
3. $\dfrac{2^{-3}}{2}$.
4. $2^{-3} \cdot 3^2$.
5. $5^{-2} \cdot 2^{-3}$.
6. $\dfrac{3^2 \cdot 3^{1.5}}{3}$.
7. $\dfrac{8^{2/3} \cdot 8^{1/3}}{8^{4/3}}$.
8. $\dfrac{5^{3/2} \cdot 5^{2/3} \cdot 5^{-1}}{5^{1/6}}$.
9. $10^4 \cdot 10^2 \cdot 10^{-2}$.
10. $\dfrac{10^7 \cdot 10^4}{10^{15} \cdot 10^{-3}}$.

6-4. Powers of 10. Values of certain powers of 10 are listed in Table 6-1 along with equivalent prefixes and abbreviations which form a part of electrical nomenclature. The advantages of employing the notation of 10 raised to a power, in place of the conventional decimal notation, are twofold: (1) the economy of space in writing (compare 10^{-12} with 0.000000000001) and (2) the facility of operating with powers of 10 in computations as a consequence of our decimal number system and of the Laws of Exponents.

The prefixes and their abbreviations are employed as exemplified by the terms: megohm (MΩ), kilocycle (kc), centimeter (cm), milliampere

TABLE 6-1. POWERS OF 10

Number	Value	Prefix	Abbreviation
10^6	1,000,000.	mega	M
10^3	1,000.	kilo	k
10^{-2}	0.01	centi	c
10^{-3}	0.001	milli	m
10^{-6}	0.000001	micro	μ
10^{-12}	0.000000000001	micromicro	$\mu\mu$

(ma), microvolt (μv), and micromicrofarad ($\mu\mu$f). A megohm is a million ohms. A kilocycle is a thousand cycles. A centimeter is a hundredth of a meter. A milliampere is a thousandth of an ampere. A microvolt is a millionth of a volt. And a micromicrofarad is a trillionth (millionth of a millionth) of a farad.

To express 200 megohms in terms of ohms, we may write

$$\begin{aligned}
200 \text{ megohms} &= 200 \cdot (1 \text{ megohm}),\\
&= 200 \cdot (10^6 \text{ ohms}) \text{ (on replacing 1 megohm by its equivalent, } 10^6 \text{ ohms)},\\
&= 2 \cdot 10^2 \cdot 10^6 \text{ ohms},\\
&= 2 \cdot 10^8 \text{ ohms}.
\end{aligned}$$

To express 50 microamperes in terms of milliamperes, we may write

$$\begin{aligned}
50 \text{ microamperes} &= 50 \cdot (1 \text{ microampere}),\\
&= 50 \cdot (10^{-6} \text{ ampere}),\\
&= 50 \cdot 10^{-3} \cdot (10^{-3} \text{ ampere}),\\
&= 50 \cdot 10^{-3} \cdot (1 \text{ milliampere}),\\
&= 5 \cdot 10^{-2} \text{ milliampere}.
\end{aligned}$$

Powers of 10 are frequently used to indicate the number of significant figures in a quantity. Thus, if we had desired to indicate that the value of $2 \cdot 10^8$ ohms is correct to three significant figures, we might have written any one of the following: $2.00 \cdot 10^8$, $20.0 \cdot 10^7$, $200 \cdot 10^6$. In this scheme the first factor includes all the significant figures, and the other factor serves only to designate the position of the decimal point.

Exercise 6–4

A. Express as a product of two factors, one of which is a power of 10:

1. 200.
2. 10.
3. 475,000.
4. 0.1
5. −0.022.
6. 0.000536.

B. Perform the indicated operations:

1. $\dfrac{60 \cdot 10^{16}}{15 \cdot 10^{17}}$.
2. $\dfrac{3 \cdot 10^{3}}{4 \cdot 10^{-}}$.

C. Convert:

1. 5 henrys to millihenrys.
2. 300 megacycles to kilocycles.
3. 250 micromicrofarads to farads.

D. Specify the number of significant figures indicated in each of the following:

1. $286 \cdot 10^{3}$.
2. $2.860 \cdot 10^{3}$.
3. $2.8600 \cdot 10^{-4}$.
4. $2.86 \cdot 10^{2}$.
5. $28.6 \cdot 10^{1}$.
6. $286 \cdot 10^{0}$.
7. $1.7 \cdot 10^{6}$.
8. $3.900 \cdot 10^{-12}$.

E. Inductive reactance, X_L, in ohms is given by the relation

$$X_L = 2\pi fL,$$

where f is the frequency in cycles per second and L is the inductance in henrys.

1. Compute the inductive reactance at 1 kc and at 15 kc for a choke which has an inductance of 10 h.
2. Compute the inductive reactance at 10 kc and at 100 Mc for a choke which has an inductance of 1700 μh.

F. Capacitive reactance, X_C, in ohms is given by the relation

$$X_C = \frac{1}{2\pi fC},$$

where f is the frequency in cycles per second and C is the capacitance in farads.

1. Compute the capacitive reactance of a 0.0025 μf condenser at 10 kc and at 10 Mc.
2. Find the difference in capacitive reactance at 1 Mc corresponding to two settings of a variable condenser, one of 250 $\mu\mu$f and the other of 300 $\mu\mu$f, respectively.

6–5. Limit. To define what is meant by a number raised to an irrational power we must introduce the fundamental concept of *limit.* Examples of sequences in which a limit is approached are the following:

$$\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \cdots.$$
$$0.3, 0.33, 0.333, 0.3333, \cdots.$$

The set of values in the first sequence above approaches the limit 1; in the second sequence the values approach $\frac{1}{3}$. Examples of sequences in which no limit is approached are the following:

$$1, 2, 3, 4, \cdots.$$
$$1, 0, 1, 0, 1, 0, \cdots.$$

By way of definition we say that a variable v approaches a limit l if the successive values of v approach l in such a manner that the numerical

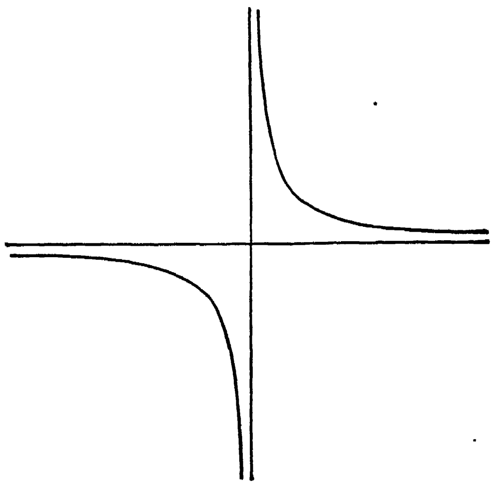


Fig. 6–1. Graph of $y = \dfrac{1}{x}\cdot$

value of the difference, $v - l$, ultimately becomes and remains less than any preassigned positive number.

In the first-quadrant branch of the graph of $y = \dfrac{1}{x}$ shown in Fig. 6–1 the ordinate of the curve approaches the limit zero as we consider points on the curve at successively increasing abscissas. In accordance with

the definition of limit: as x increases indefinitely, the quantity $|y - 0|$ ultimately becomes and remains less than any preassigned positive number. If we choose $\frac{1}{1,000,000}$ for the preassigned number, it is necessary only to continue along the curve to the right until we shall ultimately reach an abscissa beyond which $|y - 0| < \frac{1}{1,000,000}$ for every value of y. Likewise, if we choose any other number, n, we shall ultimately find $|y - 0| < n$ for all points on the curve beyond a certain point.

One important limit is the limit which is approached by the ratio of the perimeter of a polygon, which is inscribed within a fixed circle, to the diameter of the circle as the number of sides of the polygon increases indefinitely. This limit is the number π, approximately equal to 3.14159. Values are listed below for the ratio of the perimeter of a regular polygon to the diameter of the circle in which the polygon is inscribed.

Number of sides	*Ratio of perimeter of polygon to diameter of circle*
12	3.1058
24	3.1326
48	3.1394
96	3.1410
192	3.1415

Another important limit is the limit which is approached by the quantity $\left(1 + \frac{1}{m}\right)^m$ as m increases indefinitely. This limit is the number ϵ, approximately equal to 2.71828. Values of $\left(1 + \frac{1}{m}\right)^m$ for a few values of m are listed below.

m	$\left(1 + \frac{1}{m}\right)^m$	
1	$(1 + 1)^1$	$= 2.0000$
10	$\left(1 + \frac{1}{10}\right)^{10}$	$= 2.5938$
100	$\left(1 + \frac{1}{100}\right)^{100}$	$= 2.7048$

1,000	$\left(1+\frac{1}{1000}\right)^{1000}$	$= 2.7171$
10,000	$\left(1+\frac{1}{10{,}000}\right)^{10{,}000}$	$= 2.7182$

6–6. Irrational Exponents. If γ is an irrational number, then the notation a^γ means the limit approached by a^x as x approaches γ through a sequence of rational numbers. Thus, 2^π means the limit approached by the sequence of values: $2^{3.1058}$, $2^{3.1326}$, $2^{3.1394}$, $2^{3.1410}$, $2^{3.1415}$ . . . . (The powers of 2 in this sequence are the successive ratios of polygon perimeter to circle diameter which were tabulated in Sec. 6–5 in the consideration of π.)

It can be shown, although it is beyond the scope of this work to do so, that the Laws of Exponents hold for irrational as well as for rational exponents.

Exercise 6–5

Simplify:

1. $\sqrt[3]{a^\pi} \div \sqrt{a^\pi}$.
2. $\dfrac{a^2 \cdot a^{2\pi}}{a^{-3}}$.
3. $\dfrac{2^{5+\epsilon}}{2^3}$.

CHAPTER 7

QUADRATIC EQUATIONS. SQUARE ROOT

7–1. Solution of Quadratic Equations. An equation of the form

$$Ax^2 + Bx + C = 0 \tag{7-1}$$

is referred to as *a second degree,* or *quadratic, equation.* An equation of this type has for its solutions

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{7-2}$$

and

$$x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{7-3}$$

These solutions may be verified by substituting for x in Eq. (7–1) each of the values of x as given by Eqs. (7–2) and (7–3).

On substituting the value of x given by Eq. (7–2) into Eq. (7–1), we obtain

$$A \cdot \left[\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right]^2 + B \cdot \left[\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right] + C,$$

$$= A \cdot \left[\frac{B^2 - 2B\sqrt{B^2 - 4AC} + B^2 - 4AC}{4A^2}\right] + \left[\frac{-B^2 + B\sqrt{B^2 - 4AC}}{2A}\right] + C,$$

$$= \frac{AB^2 - 2AB\sqrt{B^2 - 4AC} + AB^2 - 4A^2C}{4A^2}$$

$$+ \frac{-2A^2B^2 + 2AB\sqrt{B^2 - 4AC}}{4A^2} + \frac{4A^2C}{4A^2},$$

$$= \frac{AB^2 - 2AB\sqrt{B^2 - 4AC} + AB^2 - 4A^2C - 2A^2B^2 + 2AB\sqrt{B^2 - 4AC} + 4A^2C}{4A^2},$$

$$= \frac{0}{4A^2} = 0.$$

Thus, we have shown that the value of x given by Eq. (7-2) satisfies Eq. (7-1) or, in other words, that this value of x is a solution of Eq. (7-1). In a similar manner the value of x given by Eq. (7-3) may be shown to be a solution of Eq. (7-1).

As an illustration of Eqs. (7-2) and (7-3) let us find the solutions of the equation

$$x^2 + 4x + 3 = 0.$$

By Eq. (7-2) we have

$$\begin{aligned} x &= \frac{-4 + \sqrt{16 - 12}}{2}, \\ &= \frac{-4 + 2}{2} = \frac{-2}{2} = -1. \end{aligned}$$

And by Eq. (7-3) we have

$$\begin{aligned} x &= \frac{-4 - \sqrt{16 - 12}}{2}, \\ &= \frac{-4 - 2}{2} = \frac{-6}{2} = -3. \end{aligned}$$

As a check on the numerical work we may substitute first -1 for x and then -3 for x in the original equation, $x^2 + 4x + 3 = 0$. Substituting -1 for x, we obtain

$$(-1)^2 + 4x(-1) + 3 = 1 - 4 + 3 = 0.$$

And substituting -3 for x, we obtain

$$(-3)^2 + 4x(-3) + 3 = 9 - 12 + 3 = 0.$$

In the event that $B = 0$, Eq. (7-1) reduces to

$$Ax^2 + C = 0,$$

which may be solved simply as follows:

$$\begin{aligned} Ax^2 &= -C; \\ x^2 &= -\frac{C}{A}. \end{aligned}$$

The solutions are

$$x = \sqrt{-\frac{C}{A}},$$

and

$$x = -\sqrt{-\frac{C}{A}}.$$

For the present we give no interpretation to the solution of a quadratic equation which involves the square root of a negative number. We shall consider this matter in a later chapter (Sec. 15-11).

Exercise 7-1

A. Verify that the value of x given by Eq. (7-3) is a solution of Eq. (7-1).
B. Solve the following equations, and check the result in each instance:

1. $x^2 + 5x + 4 = 0.$
2. $x^2 - 3x - 10 = 0.$
3. $x^2 - 25 = 0.$
4. $3z^2 - 48 = 0.$
5. $9t^2 + 12t + 4 = 0.$
6. $4u^2 + 8u + 3 = 0.$

7-2. Square Root; Approximation Method. The square root of any number may be obtained by a reasonable guess together with one or more divisions. To illustrate the method let us find the square root of 1000. The square root of 900 is 30, and the square root of 1600 is 40. These figures suggest that a plausible estimate of the square root of 1000 is 32.

Division of 1000 by 32 yields, to three figures, 31.2. Our estimate of 32 as the square root of 1000 was evidently high. Let us split the difference between 32 and 31.2, and as a second approximation try 31.6.

Division of 31.6 into 1000 is facilitated by our knowledge of the fact that the quotient has for its first two digits 3 and 1. The quotient to four figures comes out to be 31.64. Again splitting the difference, this time between 31.6 and 31.64, we obtain 31.62 as the square root of 1000. This process may be continued to whatever degree of accuracy is desired.

It is not essential that the first guess at the square root be a particularly good one. If we had chosen the obviously wrong value of 30 as a first estimate, we would have obtained 33.3 on division and 31.7 on averaging 30 and 33.3. A second division and averaging would then have yielded 31.62 as before. Even if we had chosen the absurd value of 40 as a first estimate, after twice dividing and averaging we would have obtained 31.64.

This method has the advantage over the conventional method (Sec. 7-3) in that it is (1) easier than the conventional method to remember, and (2) easier than the conventional method to reconstruct if it is ever forgotten.

Exercise 7-2

By the method of Sec. 7-2 obtain each of the following square roots to three figures:

1. $\sqrt{1103}$. 2. $\sqrt{242}$. 3. $\sqrt{2000}$.

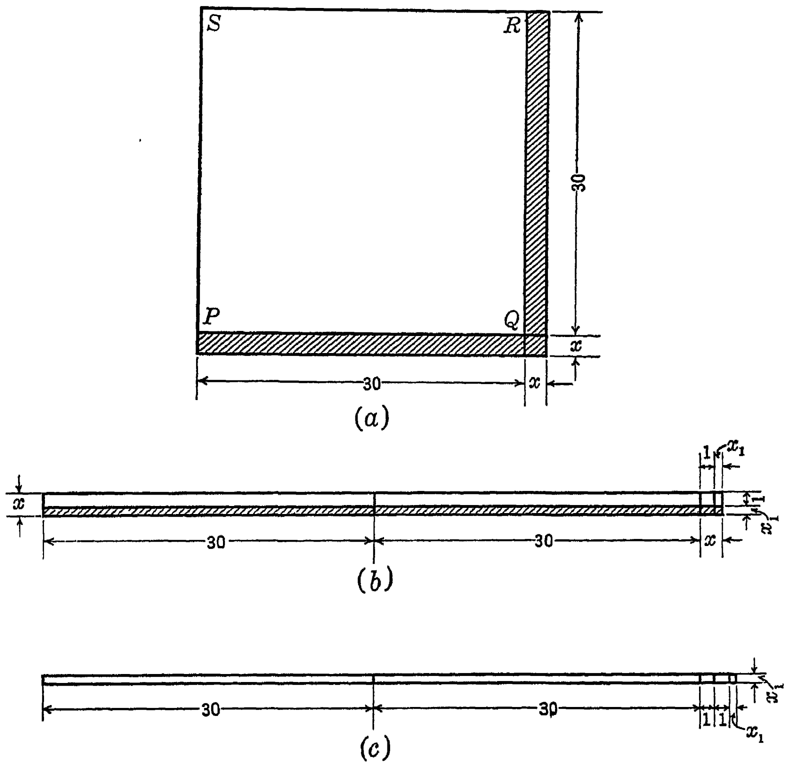


FIG. 7-1. Steps in obtaining the square root of 1000.

7-3. Square Root; Conventional Method. The steps in the conventional method of extracting a square root are illustrated in Fig. 7-1. The large outside square of Fig. 7-1(a) contains 1000 square units, and the length

of a side of this square is equal to the square root of 1000. The largest contained square whose side length is a *multiple of 10* is *PQRS*, 30 units on a side. The difference between 30 and the square root of 1000 is designated by x in Fig. 7–1(a). At this point, then, we have $\sqrt{1000} = 30 + x$, where $x < 10$. The problem is now to compute x.

Since the area *PQRS* is 900 square units, the cross-hatched border strip of Fig. 7–1(a), $x^2 + 2 \cdot 30x$, is equal to $1000 - 900$, or 100, square units. By trial we find that 1 is the largest *integral* value of x which is such that $x(x + 2 \cdot 30)$ is less than 100. The cross-hatched border strip of Fig. 7–1(a) is laid out in rectangular form in Fig. 7–1(b), where x is shown as equal to $1 + x_1$. At this point we have $\sqrt{1000} = 31 + x_1$, where $x_1 < 1$. And the problem now is to compute x_1.

The unshaded area of Fig. 7–1(b) is 61 square units; and inasmuch as the total area of Fig. 7–1(b) is 100 square units, the shaded area is $100 - 61 = 39$ square units. This 39 square unit border strip of Fig. 7–1(b) is laid out in rectangular form in Fig. 7–1(c). The evaluation of x_1 now proceeds in analogous manner to the foregoing evaluation of x. In Fig. 7–1(c) we have $39 = x_1(x_1 + 2 \cdot 31)$. By trial we find that 0.6 is the largest *tenth* value of x_1 which is such that $x_1(x_1 + 2 \cdot 31)$ is less than, or equal to, 39. At this point, then, we can say that $\sqrt{1000} = 31.6 + x_2$, where $x_2 < 0.1$.

The process may be continued to extend the evaluation of the square root to any desired number of decimal places.

The numerical work is usually arranged for simplicity as follows:

$$\begin{array}{rr}
 & 3\ \ 1.\ 6\ \ 2 \\
3 & \sqrt{10'00.00'00} \\
 & \underline{9} \\
61 & 1\ 00 \\
 & \underline{61} \\
626 & 39\ 00 \\
 & \underline{37\ 56} \\
6322 & 1\ 44\ 00 \\
 & 1\ 26\ 44
\end{array}$$

Exercise 7–3

By the method of Sec. 7–3 obtain to three figures each of the square roots indicated in Exercise 7–2.

7–4. Square Root; Special Case. Eq. (6–5), $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, facilitates the extraction of square roots in those cases wherein the number whose square root is desired may be expressed as the product of factors each of whose square root is known or is readily obtained. Thus,

$$\sqrt{288} = \sqrt{144 \cdot 2} = \sqrt{144} \cdot \sqrt{2} = 12 \cdot \sqrt{2}.$$

and

$$\sqrt{600} = \sqrt{100 \cdot 3 \cdot 2} = \sqrt{100} \cdot \sqrt{3} \cdot \sqrt{2} = 10 \cdot \sqrt{3} \cdot \sqrt{2}.$$

The square roots of 2, 3, and 5 are listed herewith for reference:

$$\sqrt{2} = 1.414. \qquad \sqrt{3} = 1.732. \qquad \sqrt{5} = 2.236.$$

Exercise 7–4

By the method of Sec. 7–4 obtain each of the following square roots to three figures:

1. $\sqrt{48}$.

2. $\sqrt{20}$.

3. $\sqrt{45}$.

4. $\sqrt{1000}$.

CHAPTER 8

SIMULTANEOUS EQUATIONS

8–1. Pairs of Linear Equations. The graphs of the lines $x - y = 1$ and $x + 2y = 4$ are drawn in Fig. 8–1. Since the coordinates of each point on the line $x - y = 1$ satisfy the relation $x - y = 1$, and the coordinates of each point on the line $x + 2y = 4$ satisfy the relation $x + 2y = 4$, it follows that the coordinates of the point of intersection of the two lines must necessarily satisfy both relations. Thus, the problem of finding the point of intersection of the two lines in Fig. 8–1 is at the same time the problem of finding a single pair of values of x and y which simultaneously satisfy both equations $x - y = 1$ and $x + 2y = 4$. Such a problem frequently arises in practice when two conditions are imposed at the same time on two variable quantities. One method of solution — the *graphical method* — consists of ascertaining the coordinates of the point of intersection of the two lines by inspection of their graphs. With reference to Fig. 8–1 we graphically find for the solution of the simultaneous equations $x - y = 1$ and $x + 2y = 4$:

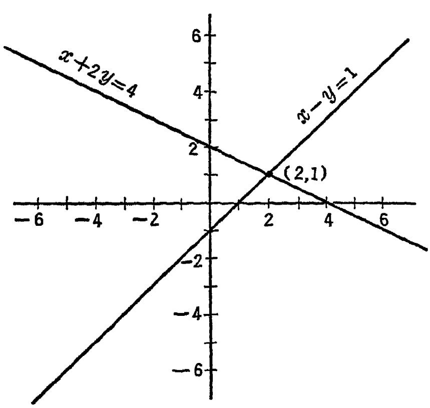


FIG. 8–1. Intersecting lines, $x - y = 1$ and $x + 2y = 4$.

$$x = 2 \quad \text{and} \quad y = 1.$$

The *analytical method* of solution, dealing with the equations rather than with their graphs, admits of several different approaches. In any analytical treatment the aim is to obtain equations involving only a single variable which are then readily solvable by methods at hand.

One analytical method of solution of the simultaneous equations

$x - y = 1$ and $x + 2y = 4$ is as follows. The equation $x - y = 1$ requires that $y = x - 1$, while the equation $x + 2y = 4$ requires that $y = 2 - \frac{x}{2}$. The two equations regarded simultaneously then impose the restriction on x:

$$x - 1 = 2 - \frac{x}{2}.$$

This is an equation in the one variable x for which the solution is found to be

$$x = 2.$$

The corresponding value of y, $y = 1$, follows immediately by the substitution of 2 for x in either of the two original equations. This method of solution is called the method of *comparison*.

The same problem might have been solved by subtracting the two given equations:

$$\begin{array}{r} x - y = 1 \\ x + 2y = 4 \\ \hline -3y = -3 \\ y = 1. \end{array}$$

Since $x + 2y$ is equal to 4, the subtraction of $x + 2y$ from the left side of the equation $x - y = 1$ and the subtraction of 4 from the right side of the equation $x - y = 1$ amount to the subtraction of the same quantity from both sides of the equation $x - y = 1$. The result, $-3y = -3$ or $y = 1$, is, then, an equation which is equivalent to $x - y = 1$ under the condition that simultaneously $x + 2y = 4$. On the substitution of 1 for y in either of the two given equations we obtain

$$x = 2.$$

Instead of subtracting the two original equations to eliminate x, we might have multiplied the first equation by 2 and then added the equations to eliminate y. Thus,

$$\begin{array}{r} 2x - 2y = 2 \\ x + 2y = 4 \\ \hline 3x = 6 \\ x = 2. \end{array}$$

The multiplication of the first given equation by 2 is here essential in order to obtain an equivalent equation in which the coefficient of y is numerically equal to the coefficient of y in the second given equation. On addition the terms in y vanish. This method of solution together with the similar preceding one is referred to as the method of *addition or subtraction.*

Another scheme is to solve one of the given equations for one variable in terms of the other variable. The resulting expression is then substituted into the second equation. For the particular problem under consideration we obtain, on solving the first given equation for x in terms of y:

$$x = 1 + y.$$

Substituting this value of x into the second given equation, we get an equation in the one variable y:

$$(1 + y) + 2y = 4,$$

which reduces to

$$3y = 3,$$
$$y = 1.$$

This method is known as the method of *substitution.*

Exercise 8–1

A. Solve the following systems of equations both graphically and analytically:

1. $x + 2y = 4;$
$x - 2y = 4.$

2. $y - x = 1;$
$x = 2.$

3. $3x - y = 7;$
$2x + y = 8.$

4. $x + 3y = 7;$
$2x - 3y = 13.$

B. For each of the following pairs of points find the equation of the line which is determined by these points:

1. (1,3), (2,5). Observe that the equation of the line,

$$Ax + By + C = 0,$$

or

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0,$$

must be satisfied by $x = 1$, $y = 3$, and by $x = 2$, $y = 5$. Hence,

$$\frac{A}{C}\cdot 1 + \frac{B}{C}\cdot 3 + 1 = 0;$$
$$\frac{A}{C}\cdot 2 + \frac{B}{C}\cdot 5 + 1 = 0.$$

It remains only to solve these two equations for $\frac{A}{C}$ and for $\frac{B}{C}$.

2. (1,1), (2,0).

8-2. Systems Involving Non-linear Equations. In case one or both of a pair of simultaneous equations are non-linear, either the method of substitution or the graphical method is suitable. In many cases the graphical method is the only practical one because of the complexities of the analytical expressions involved.

Let us consider the simultaneous equations $4x - 3y = 0$ and $x^2 + y^2 = 25$. The graphs of the two equations are plotted in Fig. 8-2, where it is seen by inspection that there are two solutions: $x = 3$, $y = 4$, and $x = -3$, $y = -4$.

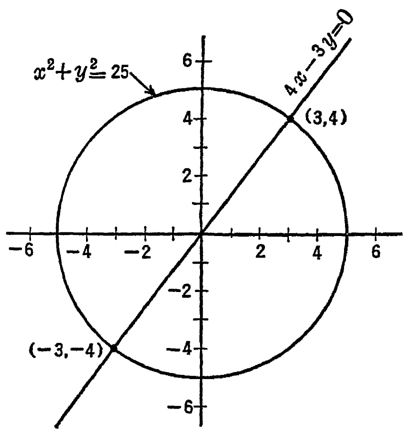


FIG. 8-2. Graphical solution of the simultaneous equations $4x - 3y = 0$ and $x^2 + y^2 = 25$.

For an analytical solution of the same problem, by the method of substitution, we solve the first equation for y to find

$$y = \frac{4}{3}x.$$

Then substituting this value of y, namely, $\frac{4}{3}x$, into the second equation, we get

$$x^2 + \frac{16}{9}x^2 = 25.$$

This is an equation in the one variable x.

Solving for x, we get

$$\frac{9x^2 + 16x^2}{9} = 25;$$
$$25x^2 = 25 \cdot 9;$$
$$x^2 = 9;$$
$$x = \pm 3.$$

The corresponding values of y follow on the substitution of 3 and -3 for x in either of the original equations.

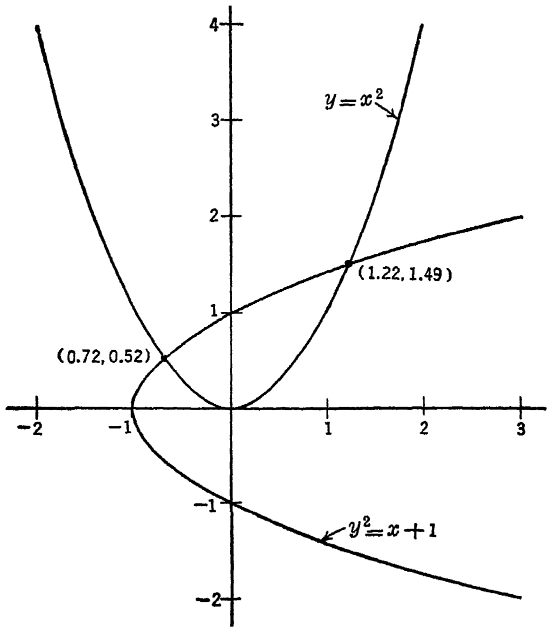


FIG. 8–3. Graphical solution of the simultaneous equations $y = x^2$ and $y^2 = x + 1$.

For another example let us consider the system: $y = x^2$ and $y^2 = x + 1$. Substituting the value of y from the first equation into the second equation, we obtain an equation in the one variable x:

$$x^4 = x + 1,$$

which is, however, an equation of the fourth degree not solvable by analytical means at our disposal. Under these circumstances we necessarily resort to graphing. The graphs of the equations are plotted in Fig. 8–3

from which the solutions are found to be $x = -0.72$, $y = 0.52$ and $x = 1.22$, $y = 1.49$.

Exercise 8–2

A. Without recourse to graphs select from the following points those points which are intersections of the graphs of $y = x + 4$ and $y = \frac{x^2}{2}$: (0,0), (−2,2), (2,6), (1,−3), (4,8), (1,$\frac{1}{2}$).

B. Solve the following systems of equations:

1. $xy = 16;$
   $y = x.$
2. $4x^2 + y^2 = 4;$
   $x = 1.$
3. $y = \frac{4}{x};$
   $x - 2y = 2.$
4. $x^2 - 2x + y^2 - 9 = 0;$
   $2x + 3y = 1.$
5. $y = x^3;$
   $y^2 = 4x.$
6. $9x^2 - 4y^2 = 36;$
   $x^2 + y^2 = 16.$

8–3. Determinants. By any of the methods of Sec. 8–1, the pair of linear equations,

$$A_1x + B_1y = C_1, \tag{8–1}$$

$$A_2x + B_2y = C_2, \tag{8–2}$$

is found to have the solution:

$$x = \frac{B_2C_1 - B_1C_2,}{A_1B_2 - A_2B_1} \tag{8–3}$$

$$y = \frac{A_1C_2 - A_2C_1.}{A_1B_2 - A_2B_1} \tag{8–4}$$

We find it useful to rewrite Eqs. (8–3) and (8–4) in the form

$$x = \frac{\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}, \tag{8–5}$$

$$y = \frac{\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}, \tag{8–6}$$

with the understanding that within each array [within each numerator and denominator in Eqs. (8-5) and (8-6)]: (1) we shall multiply in pairs those terms which occupy positions corresponding to the paired terms of Fig. 8-4; (2) we shall prefix a negative sign to a product of terms which correspond to those joined by the dotted line in Fig. 8-4. The values of x and y given by Eqs. (8-5) and (8-6) are then identical with those given by Eqs. (8-3) and (8-4).

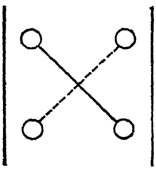


FIG. 8-4. Paired factors in Eqs. (8-5) and (8-6).

Each square array of coefficients in Eqs. (8-5) and (8-6) is referred to as a *determinant* and, in particular, as a determinant of the second *order*, since it contains two *rows* (horizontal sets) and two *columns* (vertical sets) of elements. Let us designate by Δ the particular determinant which appears in the denominator of the expressions for both x and y in Eqs. (8-5) and (8-6). We see that Δ is formed from the coefficients of x and y just as these coefficients are arranged in Eqs. (8-1) and (8-2). The numerator determinant in the expression for x in Eq. (8-5) differs formally from Δ only in that the constants C_1 and C_2 replace the coefficients of x: A_1 and A_2, respectively. The numerator determinant in the expression for y in Eq. (8-6) differs formally from Δ only in that the constants C_1 and C_2 replace the coefficients of y: B_1 and B_2, respectively.

Example. Solve the system of equations:

$$\begin{aligned} 4x + 2y &= 5, \\ 2x - y &= -3. \end{aligned}$$

Using Eqs. (8-5) and (8-6),

$$x = \frac{\begin{vmatrix} 5 & 2 \\ -3 & -1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{[5 \cdot (-1)] - [(-3) \cdot 2]}{[4 \cdot (-1)] - [2 \cdot 2]} = \frac{-5 + 6}{-4 - 4} = \frac{1}{-8} = -\frac{1}{8};$$

$$y = \frac{\begin{vmatrix} 4 & 5 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{[4 \cdot (-3)] - [2 \cdot 5]}{-8} = \frac{-12 - 10}{-8} = \frac{-22}{-8} = \frac{11}{4}.$$

Exercise 8-3

A. Derive Eqs. (8-3) and (8-4) from Eqs. (8-1) and (8-2).

B. Show that Eqs. (8-5) and (8-6) are identical with Eqs. (8-3) and (8-4), respectively.

C. Solve each of the following systems by the method of determinants:

1. $2x + 3y = 1;$
   $5x - y = 0.$
2. $\frac{1}{2}x + 7y = 0;$
   $-3x - 2y = 4.$
3. $2.3y + 1.7x - 5.5 = 0;$
   $0.8x - 1.2y = 0.$
4. $\sqrt{3}x + \sqrt{2}y = 2;$
   $\sqrt{2}x + \sqrt{3}y = 5.$

8–4. Systems of *n* Linear Equations in *n* Variables. Any equation of the type

$$A_1x_1 + A_2x_2 + A_3x_3 + \cdots + A_nx_n = B,$$

where $x_1, x_2, \cdots x_n$ are n distinct variables, is commonly referred to as a linear equation (although it is only when such an equation is limited to two variables that it is a true linear equation in the sense that it represents a line). The solution of systems of three simultaneous linear equations in three variables and the solution of n simultaneous linear equations in n variables, where n is any positive integer, are treated by the method of determinants in Chapter 29. It is in the solution of systems of more than two equations and variables that the method of determinants is particularly advantageous. (See Exercise 29–4.)

8–5. Practical Problems. Exercise 8–4 includes miscellaneous practical communications problems which are solvable by methods which have been studied up to this point in the text.

Exercise 8–4

1. The four resistances of a Wheatstone bridge circuit at balance bear the relation to each other:

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}.$$

At balance:

(a) Given $R_1 = 250$ ohms, $R_2 = 185$ ohms, and $R_3 = 210$ ohms; find R_4.
(b) Given that $R_4 = 2.52$ ohms and that $R_3 = 1.40R_1$; find R_2.

2. The charge on the electron is $4.80 \cdot 10^{-10}$ statcoulomb. Find the value of the electronic charge in coulombs, using the relation:

$$1 \text{ coulomb} = 3.00 \cdot 10^9 \text{ statcoulombs}.$$

3. The kinetic energy of a particle of mass m grams moving with a velocity v centimeters per second is $\frac{1}{2}mv^2$. The kinetic energy acquired by an electron in falling through a potential difference of V statvolts is Ve, where e is the electronic charge in statcoulombs. An electron which is accelerated through a

potential V then acquires a velocity v which is given by the relation:

$$\tfrac{1}{2}mv^2 = Ve.$$

Compute v in centimeters per second and in miles per second for an electron which is accelerated through a potential of 10,000 volts. The mass of an electron is $9.11 \cdot 10^{-28}$ gram; the charge on an electron is $4.80 \cdot 10^{-10}$ statcoulomb; 2.54 centimeters = 1 inch.

4. Consider a voltmeter of internal resistance R and an ammeter of internal resistance r connected in a circuit with a resistance ρ as shown in Fig. 8–5(a). If the resistance ρ is to be determined from the relation

$$\text{resistance} = \frac{E}{I}, \tag{8–7}$$

there is an error introduced by the fact that the voltmeter reads the potential across the series combination of resistor and ammeter instead of the desired potential across the resistor alone. That resistance which is determined by

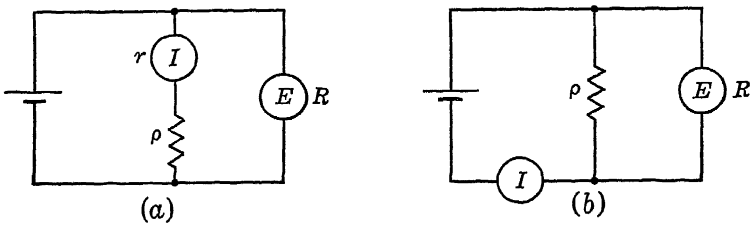


FIG. 8–5. Resistance measurement by voltmeter-ammeter method.

Eq. (8–7) is the resistance $r + \rho$ of the series combination, and the error introduced in using this value is the difference between $r + \rho$ and ρ, that is, r. The relative error is $\frac{r}{\rho}$, the ratio of the error in the measurement, r, and the true value of the resistance, ρ. This analysis shows that if a resistance is to be measured with the arrangement of Fig. 8–5(a), best accuracy will be obtained when the resistance to be measured is large compared with that of the ammeter.

Investigate the nature of the error introduced by the arrangement of Fig. 8–5(b), and indicate for what order of magnitude of resistance, ρ, this method is suitable. Here the ammeter reads the sum of the currents through ρ and R instead of simply the desired current through ρ. The resistance which is actually measured is, then, R_0, the resistance of ρ and R in parallel, where R_0 is given by the relation

$$\frac{1}{R_0} = \frac{1}{\rho} + \frac{1}{R}.$$

5. A generator of emf E and internal resistance r is connected to a load of resistance mr. The power developed in the load is given by

$$P = \left[\frac{E}{r + mr}\right]^2 \cdot mr. \tag{8–8}$$

(a) By algebraic manipulations show that Eq. (8–8) becomes

$$P = \frac{E^2}{r} \cdot \frac{1}{\dfrac{(1 - m)^2}{m} + 4}. \tag{8–9}$$

(b) From Eq. (8–9) show that the power is a maximum when $m = 1$, that is, when the resistance of the load is equal to the internal resistance of the generator.

(c) Show that the maximum power is given by

$$P_m = \frac{E^2}{4r}.$$

6. For three resistances R_1, R_2, and R_3 in parallel, the net resistance is given by R, where

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Show that

$$R = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1R_2R_3}.$$

7. In a degenerative amplifier a fraction, F, of the output voltage is fed back to the input so as to oppose the applied input voltage. If the applied input voltage is e, then the net input voltage is $e - FE$. The output voltage is equal to the net input voltage multiplied by the voltage amplification, A, of the amplifier. Thus,

$$E = A(e - FE). \tag{8–10}$$

(a) Show from Eq. (8–10) that the effective amplification, which is the ratio of the output voltage, E, to the applied input voltage, e, is given by

$$\frac{E}{e} = \frac{A}{1 + FA}. \tag{8–11}$$

(b) Show from Eq. (8–11) that if FA is sufficiently large, the voltage amplification becomes substantially independent of A. (Physically, this means that the degenerative amplifier system is very stable, providing essentially uniform effective voltage amplification despite possible changes in

operating conditions which might appreciably alter A, the voltage amplification of the amplifier itself.)

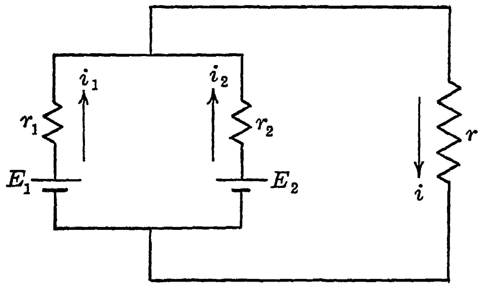


FIG. 8-6. Parallel batteries feeding resistive load.

8. In the circuit of Fig. 8-6 Kirchhoff's laws yield:

$$i = i_1 + i_2 \tag{8-12}$$

$$E_1 = i_1 r_1 + ir \tag{8-13}$$

$$E_2 = i_2 r_2 + ir \tag{8-14}$$

Show from Eqs. (8-12), (8-13), and (8-14) that

$$i_1 = \frac{E_1(r + r_2) - E_2 r}{(r_1 + r)(r_2 + r) - r^2}.$$

CHAPTER 9

TRIGONOMETRIC FUNCTIONS

9–1. Angles. An angle is a measure of the rotation of a line about a fixed point. The rotating line in its initial position is called the *initial side* of the angle, and in its final position is called the *terminal side* of the angle. In general, in electrical circuit studies we shall regard the fixed point of the angle as the origin of a coordinate system, and we shall take the initial side of the angle to be coincident with the positive x-axis. We shall define the sign of an angle as *positive* if the rotation is counterclockwise, *negative* if the rotation is clockwise. Further, we shall permit angles of any magnitude, even of several revolutions. (An angle of two revolutions is 720°.) Two angles which differ by an integral multiple of 360° — and which, therefore, have the same terminal side — are said to be *coterminal.*

An *acute angle* is an angle which is numerically between 0° and 90°. An *obtuse angle* is angle which is numerically between 90° and 180°. Acute and obtuse angles are collectively referred to as *oblique angles.* Angles which are described as acute, obtuse, or oblique are usually understood to be positive.

If the sum of two angles is 90°, each angle is called the *complement* of the other. If the sum of two angles is 180°, each angle is called the *supplement* of the other.

9–2. Definition of Sine, Cosine, and Tangent. In engineering studies certain quantities related to angles occur so frequently that it is found advantageous to assign each of these quantities a name. Collectively, these quantities are referred to as *trigonometric functions.* Individually, they are called *sine, cosine, tangent, cotangent, secant,* and *cosecant.* The first three trigonometric functions, namely, the sine, cosine, and tangent, we shall presently define. The last three functions, which are the reciprocals of the first three, we shall consider separately in Sec. 9–12.

In Fig. 9–1 θ is the angle between the positive x-axis and the line OM. With any particular point P, of coordinates (x,y), on the line OM we

associate three distances: x, the abscissa, y, the ordinate, and r, the *radius vector*. The radius vector to a point is the distance from the origin

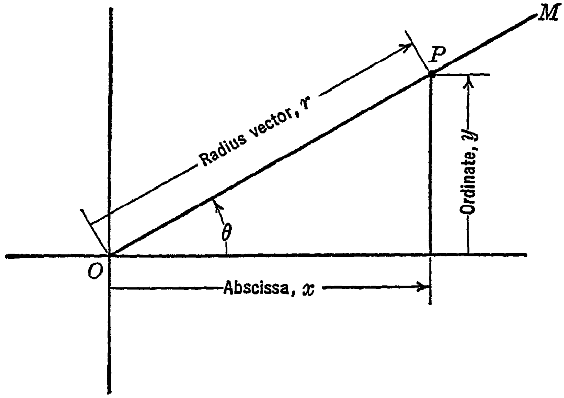


FIG. 9–1. $\sin\theta = \dfrac{y}{r}$; $\cos\theta = \dfrac{x}{r}$; $\tan\theta = \dfrac{y}{x}$.

to the point, this distance being considered always as positive regardless of the quadrant in which the point lies. We define:

$$\text{sine } \theta \text{ (abbreviated } \sin\theta) = \frac{\text{ordinate}}{\text{radius vector}} = \frac{y}{r}; \tag{9–1}$$

$$\text{cosine } \theta \text{ (abbreviated } \cos\theta) = \frac{\text{abscissa}}{\text{radius vector}} = \frac{x}{r}; \tag{9–2}$$

$$\text{tangent } \theta \text{ (abbreviated } \tan\theta) = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}. \tag{9–3}$$

The magnitudes of the ratios defined above by Eqs. (9–1) through (9–3) are independent of the particular point P which is chosen on the line OM. This is evident on inspection of Fig. 9–2 where it is seen that

$$\frac{\overline{AB}}{\overline{OB}} = \frac{\overline{CD}}{\overline{OD}} = \frac{\overline{EF}}{\overline{OF}};$$

$$\frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OC}}{\overline{OD}} = \frac{\overline{OE}}{\overline{OF}};$$

$$\frac{\overline{AB}}{\overline{OA}} = \frac{\overline{CD}}{\overline{OC}} = \frac{\overline{EF}}{\overline{OE}}.$$

Each of the ratios, $\sin\theta$, $\cos\theta$, and $\tan\theta$, is dependent only on the position of the line OM. In other words, each of these ratios is a function of θ.

The distances — abscissa, ordinate, and radius vector — which are employed in the definitions of sine, cosine, and tangent are all measured

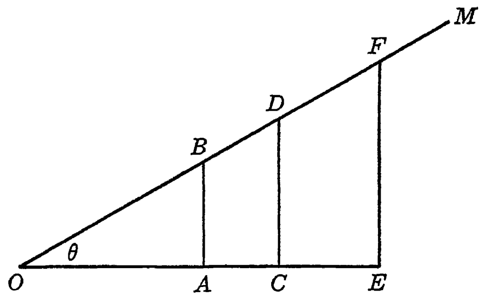


FIG. 9–2. Equal ratios between corresponding pairs of sides in similar right triangles.

in the same dimensions so that each of these trigonometric functions is dimensionless, that is, each is a pure number. To illustrate, if in Fig. 9–1 $x = 4$ inches and $y = 3$ inches, then $r = \sqrt{4^2 + 3^2}$ inches $= 5$ inches, from which:

$$\sin\theta = \frac{3 \text{ inches}}{5 \text{ inches}} = 0.6 \text{ (dimensionless)};$$

$$\cos\theta = \frac{4 \text{ inches}}{5 \text{ inches}} = 0.8 \text{ (dimensionless)};$$

$$\tan\theta = \frac{3 \text{ inches}}{4 \text{ inches}} = 0.75 \text{ (dimensionless)}.$$

Since in Fig. 9–1 it is the position of the terminal line OM which determines the values of the trigonometric functions, it is evident that each of the functions of a given angle is equal to the corresponding function of any coterminal angle. For example, each of the functions of 100° is equal to the corresponding function of 100° + 360° or of 100° − 360°

Exercise 9–1

Using a protractor and rule lay out the following angles; by measurements

of lengths evaluate sine, cosine, and tangent for each angle:

1. $20°$.
2. $40°$.
3. $80°$.
4. $100°$.
5. $200°$.
6. $320°$.
7. $380°$.
8. $740°$.
9. $-20°$.
10. $-40°$.

9–3. Special Cases. In this section we shall compute values of the trigonometric functions in certain simple cases. Since in these computations the choice of the radius vector is irrelevant, we choose to work with a radius vector which is 1 unit in length. In Fig. 9–1 with a radius vector which is 1 unit in length

$$x^2 + y^2 = 1 \tag{9–4}$$

for any angle θ.

For the particular case of $\theta = 45°$

$$x = y,$$

and Eq. (9–4) becomes

$$2x^2 = 1,$$

so that

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2} = \pm 0.707 = y.$$

The formal relations $x^2 + y^2 = 1$ and $x = y$ are satisfied by $x = \pm 0.707$, $y = \pm 0.707$. However, with reference to Fig. 9–1, we see that for the case of $\theta = 45°$ it is only the positive values of x and y with which we are concerned. Thus, for $\theta = 45°$ and with $r = 1$, $x = 0.707$ and $y = 0.707$ from which:

$$\sin 45° = \frac{y}{r} = \frac{0.707}{1} = 0.707;$$

$$\cos 45° = \frac{x}{r} = \frac{0.707}{1} = 0.707;$$

$$\tan 45° = \frac{y}{x} = \frac{0.707}{0.707} = 1.$$

For the particular case of $\theta = 30°$, $y = \frac{1}{2}$. This is apparent from Fig. 9-3 wherein the triangle OAB is one half of the equilateral triangle

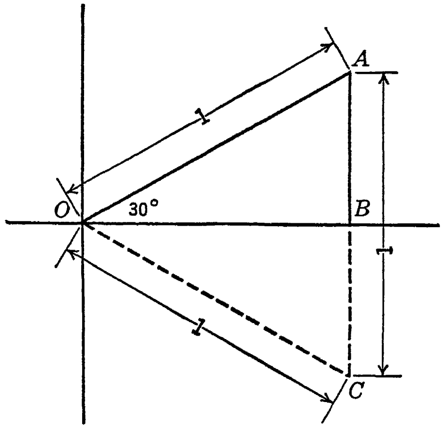


FIG. 9-3. Triangle OAB is one-half of triangle OAC.

OAC. (Each angle of an equilateral triangle equals 60°.) For $\theta = 30°$, then, Eq. (9-4) becomes

$$x^2 + \left(\frac{1}{2}\right)^2 = 1,$$

or

$$x^2 = \frac{3}{4},$$

so that

$$x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2} = \pm 0.866.$$

With reference to Fig. 9-3 we see that for $\theta = 30°$ both x and y are positive. Thus,

$$\sin 30° = \frac{y}{r} = \frac{0.5}{1} = 0.5;$$

$$\cos 30° = \frac{x}{r} = \frac{0.866}{1} = 0.866;$$

$$\tan 30° = \frac{y}{x} = \frac{0.5}{0.866} = 0.577.$$

In Fig. 9–4 the points P_1, P_2, P_3, and P_4, each of radius vector 1, are associated with the angles 30°, 150°, 210°, and 330°, respectively. The

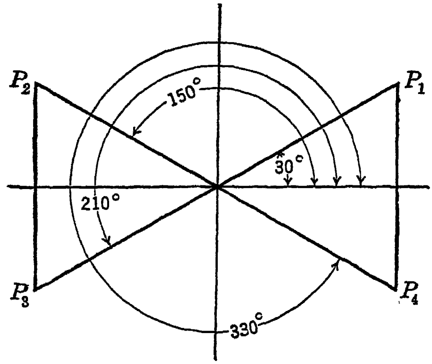


FIG. 9–4. Angles 30°, 150°, 210°, and 330°.

coordinates of these points differ only in sign; consequently, the trigonometric functions for the four angles differ only in sign:

$$\sin 150° = \frac{0.5}{1} = 0.5;$$

$$\cos 150° = \frac{-0.866}{1} = -0.866;$$

$$\tan 150° = \frac{0.5}{-0.866} = -0.577.$$

$$\sin 210° = \frac{-0.5}{1} = -0.5;$$

$$\cos 210° = \frac{-0.866}{1} = -0.866;$$

$$\tan 210° = \frac{-0.5}{-0.866} = 0.577.$$

$$\sin 330° = \frac{-0.5}{1} = -0.5;$$

$$\cos 330° = \frac{0.866}{1} = 0.866;$$

$$\tan 330° = \frac{-0.5}{0.866} = 0.577.$$

Exercise 9–2

1. Show that $\sin 60° = \cos 30° = 0.866$; $\cos 60° = \sin 30° = 0.5$; $\tan 60° = 1.732$.
2. Show that for any acute angle θ

$$\sin \theta = \cos (90° - \theta). \tag{9–5}$$

(Construct a figure for the complement of θ like Fig. 9–1 for θ. Observe similar triangle relationships.)
3. Show that Eq. (9–5) is valid for any angle whose terminal side lies in the first quadrant (for example, for such an angle as 380°).
4. Show that Eq. (9–5) is valid for *any* angle.
5. Fill in the blanks in the accompanying table. Show all computations involved.

Angle	60°	120°	240°	300°
Sine	0.866	0.866		
Cosine	0.5		−0.5	
Tangent	1.732			−1.732

6. Fill in the blanks in the accompanying table, regarding θ as an acute angle. Show all computations involved.

Angle	θ	$180° - \theta$	$180° + \theta$	
Sine	$\sin \theta$	$\sin \theta$		
Cosine	$\cos \theta$		$-\cos \theta$	
Tangent	$\tan \theta$			$-\tan \theta$

7. Show that the relations in the completed table of Problem 6 above are valid where θ is any angle.

9–4. Graphs of the Trigonometric Functions. In Sec. 9–3 we computed values of the trigonometric functions of certain angles. Similar computations can be performed to evaluate functions of angles in other simple cases. In any case the trigonometric functions of a given angle can be evaluated by the graphical method of Exercise 9–1. From such computations graphs of the trigonometric functions can be plotted as in Fig. 9–5. An appreciation of the nature of the sine, cosine, and tangent functions as exemplified by the curves of Fig. 9–5 is essential to a study of communications.

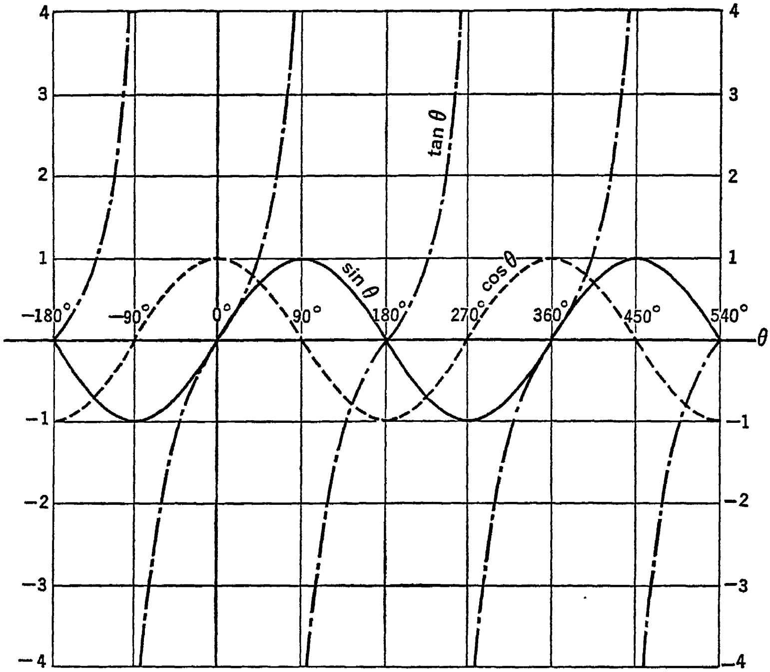


FIG. 9–5. Graphs of the trigonometric functions.

Exercise 9–3

1. Copy the curves of Fig. 9–5, and extend the range of these curves 180° to the left and 180° to the right.

2. Fig. 9–5 suggests that the cosine curve is identical with the sine curve except for a displacement of 90° along the x-axis, that is, that

$$\cos \theta = \sin (90^\circ + \theta). \tag{9–6}$$

For each of the following values of θ construct a figure and demonstrate the validity of Eq. (9–6):

a. 30°.
b. Any angle whose terminal side lies in the first quadrant.
c. Any angle whose terminal side lies in the second quadrant.
d. Any angle whose terminal side lies in the third quadrant.
e. Any angle whose terminal side lies in the fourth quadrant.

9–5. Trigonometric Functions Expressed as Functions of Acute Angles. The curves of Fig. 9–5 provide a convenient means of relating the func-

tions of any given angle with the functions of an acute angle. For specific examples, with reference to the sine curve we observe that

$$\sin 128° = \sin (90° + 38°) = \sin (90° - 38°) = \sin 52°;$$

with reference to the cosine curve we see that

$$\cos 531° = \cos (540° - 9°) = -\cos 9°;$$

and with reference to the tangent curve we see that

$$\tan 185° = \tan (180° + 5°) = \tan 5°.$$

In this way a rough sketch of the sine, cosine, and tangent curves serves adequately to relate the functions of any angle with the functions of an acute angle; it is then essential to know accurately only the functions of the acute angles in order to determine the functions of any angle.

It is possible to express the cosine of any angle in terms of the sine of an acute angle. Thus, with reference to the sine and cosine curves of Fig. 9–5, we see that

$$\cos 15° = \sin 75°;$$

and

$$\cos 530° = \cos (540° - 10°) = -\sin (90° - 10°) = -\sin 80°.$$

Many slide rules list sines of angles but not cosines; and in anticipation of using a slide rule in work with trigonometric functions it is good practice to associate both sines and cosines with *sines* of acute angles.

In place of using the curves of Fig. 9–5, various formulas and mnemonics may be employed to describe properties of the trigonometric functions, but inasmuch as the curves are in themselves basic accessories of the communications engineer, it is simplest to exploit these curves as far as possible.

Exercise 9–4

Given: $\sin 30° = 0.5$, $\sin 45° = 0.707$, $\sin 60° = 0.866$, $\tan 30° = 0.577$, $\tan 45° = 1$, $\tan 60° = 1.732$; with reference to Fig. 9–5 evaluate:

1. $\sin 135°$.
2. $\sin 150°$.
3. $\sin (-60°)$.
4. $\sin (390°)$.
5. $\cos 60°$.
6. $\cos 315°$.
7. $\cos 510°$.
8. $\cos (-45°)$.
9. $\tan 120°$.
10. $\tan 225°$.

9-6. Table of Trigonometric Functions. In Table 9-1 are given the sines, cosines, and tangents of angles between 0° and 90°. In the left column are listed angles from 0° to 45°, and in the right column are listed angles from 45° to 90°. Angles which are in the same row, for example, 0° and 90°, are complementary. Inasmuch as the sine of an angle equals the cosine of the complementary angle (Problem 2, Exercise 9-2), the second column serves both for sines from 0° to 45° and for cosines from 45° to 90°. The third column serves both for sines from 45° to 90° and for cosines from 0° to 45°.

Exercise 9-5

With the aid of Table 9-1 evaluate:

1. sin 36°.
2. sin 93°.
3. sin 125°.
4. cos 20°.
5. sin 184°.
6. cos 172°.
7. sin (−5°).
8. tan 190°.
9. tan 95°.
10. tan (−8°).

9-7. Inverse Relations. The table of trigonometric functions, Table 9-1, can be used either to find the value of a particular function for a given angle or, inversely, to find an angle corresponding to a given value of a function. Fig. 9-5 shows that there is an indefinite number of angles corresponding to each value of any of the trigonometric functions. For example, corresponding to $\cos\theta = -1$, there are, among others, the following values of θ: −180°, 180°, 540°.

In mathematical symbolism one writes $\sin^{-1} A$ for "an angle whose sine is A"; $\cos^{-1} B$ for "an angle whose cosine is B"; and $\tan^{-1} C$ for "an angle whose tangent is C." Sine^{-1}, cosine^{-1}, and tangent^{-1} are called *inverse trigonometric functions.* Sometimes sine^{-1}, cosine^{-1} and tangent^{-1} are written as arc sine, arc cosine, and arc tangent, respectively.

The symbol −1 as used here is not an exponent. If we desire to indicate $\dfrac{1}{\sin\theta}$ by means of the exponent −1, we write $(\sin\theta)^{-1}$. The nth power of $\sin\theta$ may be represented without ambiguity either by $\sin^n\theta$ or by $(\sin\theta)^n$. The notation $\sin^n\theta$ is generally favored.

TABLE 9–1. SINES, COSINES, AND TANGENTS OF ANGLES FROM 0° TO 90°

Angle 0° to 45°	Sine 0° to 45° Cosine 45° to 90°	Sine 45° to 90° Cosine 0° to 45°	Tangent 0° to 45°	Tangent 45° to 90°	Angle 45° to 90°
0° 0′	.0000	1.0000	0000	Infinite	90° 0′
10	.0029	1.0000	.0029	343.7737	50
20	.0058	1.0000	.0058	171 8854	40
30	.0087	1.0000	.0087	114.5887	30
40	.0116	.9999	.0116	85.9398	20
50	.0145	.9999	.0145	68.7501	10
1° 0′	.0175	.9998	.0175	57.2900	89° 0′
10	.0204	.9998	.0204	49.1039	50
20	.0233	.9997	.0233	42.9641	40
30	.0262	.9997	.0262	38.1885	30
40	.0291	.9996	.0291	34.3678	20
50	.0320	.9995	.0320	31.2416	10
2° 0′	.0349	.9994	.0349	28.6363	88° 0′
10	.0378	.9993	.0378	26.4316	50
20	.0407	.9992	.0407	24.5418	40
30	.0436	.9990	.0437	22.9038	30
40	.0465	.9989	.0466	21.4704	20
50	.0494	.9988	.0495	20.2056	10
3° 0′	.0523	.9986	.0524	19.0811	87° 0′
10	.0552	.9985	.0553	18.0750	50
20	.0581	.9983	.0582	17.1693	40
30	.0610	.9981	.0612	16.3499	30
40	.0640	.9980	.0641	15.6048	20
50	.0669	.9978	.0670	14.9244	10
4° 0′	.0698	.9976	.0699	14.3007	86° 0′
10	.0727	.9974	.0729	13.7267	50
20	.0756	.9971	.0758	13.1969	40
30	.0785	.9969	.0787	12.7062	30
40	.0814	.9967	.0816	12.2505	20
50	.0843	.9964	.0846	11.8262	10
5° 0′	.0872	.9962	.0875	11.4301	85° 0′
10	.0901	.9959	.0904	11.0594	50
20	.0929	.9957	.0934	10.7119	40
30	.0958	.9954	.0963	10.3854	30
40	.0987	.9951	.0992	10.0780	20
50	.1016	.9948	.1022	9.7882	10
6° 0′	.1045	.9945	.1051	9.5144	84° 0′
10	.1074	.9942	.1080	9.2553	50
20	.1103	.9939	.1110	9.0098	40
30	.1132	.9936	.1139	8.7769	30
40	.1161	.9932	.1169	8.5555	20
50	.1190	.9929	.1198	8.3450	10
7° 0′	.1219	.9925	.1228	8.1443	83° 0′
10	.1248	.9922	.1257	7.9530	50
20	.1276	.9918	.1287	7.7704	40
30	.1305	.9914	.1317	7.5958	30
40	.1334	9911	.1346	7.4287	20
50	.1363	.9907	.1376	7.2687	10
8° 0′	.1392	.9903	.1405	7.1154	82° 0′
10	.1421	.9899	.1435	6.9682	50
20	.1449	.9894	.1465	6.8269	40
30	.1478	.9890	.1495	6.6912	30
40	.1507	.9886	.1524	6.5606	20
50	.1536	.9881	.1554	6.4348	10
9° 0′	.1564	.9877	.1584	6.3138	81° 0′
10	.1593	.9872	.1614	6.1970	50
20	.1622	.9868	.1644	6.0844	40
30	.1650	.9863	.1673	5.9758	30
40	.1679	.9858	.1703	5.8708	20
50	.1708	.9853	.1733	5.7694	10
10° 0′	.1736	.9848	1763	5.6713	80° 0′
10	.1765	.9843	.1793	5.5764	50
20	.1794	.9838	.1823	5.4845	40
30	.1822	.9833	.1853	5.3955	30
40	.1851	.9827	.1883	5.3093	20
50	.1880	.9822	.1914	5.2257	79°10′

TABLE 9–1. *Continued*

Angle 0° to 45°	Sine 0° to 45° Cosine 45° to 90°	Sine 45° to 90° Cosine 0° to 45°	Tangent 0° to 45°	Tangent 45° to 90°	Angle 45° to 90°
11° 0′	.1908	.9816	.1944	5.1446	**79° 0′**
10	.1937	.9811	.1974	5.0658	50
20	.1965	.9805	.2004	4.9894	40
30	.1994	.9799	.2035	4.9152	30
40	.2022	.9793	.2065	4.8430	20
50	.2051	.9787	.2095	4.7729	10
12° 0′	.2079	.9781	.2126	4.7046	**78° 0′**
10	.2108	.9775	.2156	4.6382	50
20	.2136	.9769	.2186	4.5736	40
30	.2164	.9763	.2217	4.5107	30
40	.2193	.9757	.2247	4.4494	20
50	.2221	.9750	.2278	4.3897	10
13° 0′	.2250	.9744	.2309	4.3315	**77° 0′**
10	.2278	.9737	.2339	4.2747	50
20	.2306	.9730	.2370	4.2193	40
30	.2334	.9724	.2401	4.1653	30
40	.2363	.9717	.2432	4.1126	20
50	.2391	.9710	.2462	4.0611	10
14° 0′	.2419	.9703	.2493	4.0108	**76° 0′**
10	.2447	.9696	.2524	3.9617	50
20	.2476	.9689	.2555	3.9136	40
30	.2504	.9681	.2586	3.8667	30
40	.2532	.9674	.2617	3.8208	20
50	.2560	.9667	.2648	3.7760	10
15° 0′	.2588	.9659	.2679	3.7321	**75° 0′**
10	.2616	.9652	.2711	3.6891	50
20	.2644	.9644	.2742	3.6470	40
30	.2672	.9636	.2773	3.6059	30
40	.2700	.9628	.2805	3.5656	20
50	.2728	.9621	.2836	3.5261	10
16° 0′	.2756	.9613	.2867	3.4874	**74° 0′**
10	.2784	.9605	.2899	3.4495	50
20	.2812	.9596	.2931	3.4124	40
30	.2840	.9588	.2962	3.3759	30
40	.2868	.9580	.2994	3.3402	20
50	.2896	.9572	.3026	3.3052	10
17° 0′	.2924	.9563	.3057	3.2709	**73° 0′**
10	.2952	.9555	.3089	3.2371	50
20	.2979	.9546	.3121	3.2041	40
30	.3007	.9537	.3153	3.1716	30
40	.3035	.9528	.3185	3.1397	20
50	.3062	.9520	.3217	3.1084	10
18° 0′	.3090	.9511	.3249	3.0777	**72° 0′**
10	.3118	.9502	.3281	3.0475	50
20	.3145	.9492	.3314	3.0178	40
30	.3173	.9483	.3346	2.9887	30
40	.3201	.9474	.3378	2.9600	20
50	.3228	.9465	.3411	2.9319	10
19° 0′	.3256	.9455	.3443	2.9042	**71° 0′**
10	.3283	.9446	.3476	2.8770	50
20	.3311	.9436	.3508	2.8502	40
30	.3338	.9426	.3541	2.8239	30
40	.3365	.9417	.3574	2.7980	20
50	.3393	.9407	.3607	2.7725	10
20° 0′	.3420	.9397	.3640	2.7475	**70° 0′**
10	.3448	.9387	.3673	2.7228	50
20	.3475	.9377	.3706	2.6985	40
30	.3502	.9367	.3739	2.6746	30
40	.3529	.9356	.3772	2.6511	20
50	.3557	.9346	.3805	2.6279	10
21° 0′	.3584	.9336	.3839	2.6051	**69° 0′**
10	.3611	.9325	.3872	2.5826	50
20	.3638	.9315	.3906	2.5605	40
30	.3665	.9304	.3939	2.5386	30
40	.3692	.9293	.3973	2.5172	20
50	.3719	.9283	.4006	2.4960	**68° 10′**

TABLE 9–1. *Continued*

Angle 0° to 45°	Sine 0° to 45° Cosine 45° to 90°	Sine 45° to 90° Cosine 0° to 45°	Tangent 0° to 45°	Tangent 45° to 90°	Angle 45° to 90°
22° 0′	.3746	.9272	.4040	2.4751	**68° 0′**
10	.3773	.9261	.4074	2.4545	50
20	.3800	.9250	.4108	2.4342	40
30	.3827	.9239	.4142	2.4142	30
40	.3854	.9228	.4176	2.3945	20
50	.3881	.9216	.4210	2.3750	10
23° 0′	.3907	.9205	.4245	2.3559	**67° 0′**
10	.3934	.9194	.4279	2.3369	50
20	.3961	.9182	.4314	2.3183	40
30	.3987	.9171	.4348	2.2998	30
40	.4014	.9159	.4383	2.2817	20
50	.4041	.9147	.4417	2.2637	10
24° 0′	4067	.9135	.4452	2.2460	**66° 0′**
10	.4094	.9124	.4487	2.2286	50
20	4120	.9112	.4522	2.2113	40
30	.4147	.9100	.4557	2.1943	30
40	.4173	.9088	.4592	2.1775	20
50	.4200	.9075	.4628	2.1609	10
25° 0′	.4226	.9063	.4663	2.1445	**65° 0′**
10	.4253	.9051	.4699	2.1283	50
20	.4279	.9038	.4734	2.1123	40
30	.4305	.9026	.4770	2 0965	30
40	.4331	.9013	.4806	2 0809	20
50	.4358	.9001	.4841	2.0655	10
26° 0′	.4384	.8988	.4877	2.0503	**64° 0′**
10	.4410	.8975	.4913	2.0353	50
20	.4436	.8962	4950	2.0204	40
30	.4462	.8949	4986	2 0057	30
40	.4488	.8936	.5022	1.9912	20
50	.4514	.8923	.5059	1.9768	10
27° 0′	.4540	.8910	.5095	1 9626	**63° 0′**
10	4566	.8897	.5132	1.9486	50
20	4592	.8884	.5169	1.9347	40
30	.4617	.8870	.5206	1.9210	30
40	.4643	.8857	.5243	1.9074	20
50	.4669	.8843	.5280	1.8940	10
28° 0′	.4695	.8829	.5317	1.8807	**62° 0′**
10	.4720	.8816	.5354	1.8676	50
20	.4746	.8802	.5392	1.8546	40
30	.4772	.8788	.5430	1.8418	30
40	.4797	.8774	.5467	1.8291	20
50	.4823	.8760	.5505	1.8165	10
29° 0′	.4848	.8746	.5543	1.8040	**61° 0′**
10	.4874	.8732	.5581	1.7917	50
20	.4899	.8718	.5619	1.7796	40
30	.4924	.8704	.5658	1.7675	30
40	.4950	.8689	.5696	1.7556	20
50	.4975	.8675	.5735	1.7437	10
30° 0′	.5000	.8660	.5774	1.7321	**60° 0′**
10	5025	.8646	.5812	1.7205	50
20	.5050	.8631	.5851	1.7090	40
30	.5075	.8616	.5890	1.6977	30
40	.5100	.8601	.5930	1.6864	20
50	.5125	.8587	.5969	1.6753	10
31° 0′	.5150	.8572	.6009	1 6643	**59° 0′**
10	.5175	.8557	.6048	1.6534	50
20	.5200	.8542	.6088	1 6426	40
30	.5225	.8526	.6128	1.6319	30
40	.5250	.8511	.6168	1.6212	20
50	.5275	.8496	.6208	1.6107	10
32° 0′	.5299	.8480	.6249	1.6003	**58° 0′**
10	.5324	.8465	.6289	1.5900	50
20	.5348	.8450	.6330	1.5798	40
30	.5373	.8434	.6371	1.5697	30
40	.5398	.8418	.6412	1.5597	20
50	.5422	.8403	.6453	1.5497	10
33° 0′	.5446	.8387	.6494	1.5399	**57° 0′**
10	.5471	.8371	.6536	1.5301	50
20	.5495	.8355	.6577	1.5204	40
30	.5519	.8339	.6619	1.5108	30
40	.5544	.8323	.6661	1.5013	20
50	.5568	.8307	.6703	1.4919	**56°10′**

TABLE 9-1. *Continued*

Angle 0° to 45°	Sine 0° to 45° Cosine 45° to 90°	Sine 45° to 90° Cosine 0° to 45°	Tangent 0° to 45°	Tangent 45° to 90°	Angle 45° to 90°
34° 0′	.5592	.8290	.6745	1.4826	**56° 0′**
10	.5616	.8274	.6787	1.4733	50
20	.5640	.8258	.6830	1.4641	40
30	.5664	.8241	.6873	1.4550	30
40	.5688	.8225	.6916	1.4460	20
50	.5712	.8208	.6959	1.4370	10
35° 0′	.5736	.8192	.7002	1.4281	**55° 0′**
10	.5760	.8175	7046	1.4193	50
20	.5783	.8158	.7089	1 4106	40
30	.5807	.8141	.7133	1.4019	30
40	.5831	.8124	.7177	1.3934	20
50	.5854	.8107	.7221	1.3848	10
36° 0′	.5878	.8090	.7265	1 3764	**54° 0′**
10	.5901	.8073	.7310	1.3680	50
20	.5925	.8056	.7355	1 3597	40
30	.5948	.8039	7400	1.3514	30
40	.5972	.8021	.7445	1.3432	20
50	.5995	.8004	.7490	1.3351	10
37° 0′	.6018	7986	.7536	1.3270	**53° 0′**
10	.6041	.7969	.7581	1 3190	50
20	.6065	.7951	.7627	1.3111	40
30	.6088	.7934	.7673	1.3032	30
40	.6111	.7916	.7720	1.2954	20
50	.6134	.7898	.7766	1.2876	10
38° 0′	.6157	.7880	7813	1.2799	**52° 0′**
10	.6180	.7862	7860	1 2723	50
20	.6202	.7844	.7907	1.2647	40
30	.6225	.7826	.7954	1 2572	30
40	.6248	.7808	.8002	1.2497	20
50	.6271	.7790	.8050	1.2423	10
39° 0′	.6293	.7771	.8098	1.2349	**51° 0′**
10	.6316	.7753	.8146	1.2276	50
20	6338	7735	.8195	1.2203	40
30	6361	.7716	.8243	1.2131	30
40	6383	.7698	.8292	1.2059	20
50	.6406	.7679	.8342	1.1988	10
40° 0′	.6428	.7660	.8391	1.1918	**50° 0′**
10	.6450	7642	.8441	1.1847	50
20	6472	7623	.8491	1.1778	40
30	6494	7604	.8541	1 1708	30
40	.6517	.7585	.8591	1.1640	20
50	.6539	7566	.8642	1 1571	10
41° 0′	.6561	.7547	.8693	1 1504	**49° 0′**
10	.6583	7528	.8744	1.1436	50
20	.6604	.7509	.8796	1.1369	40
30	.6626	.7490	.8847	1 1303	30
40	6648	.7470	.8899	1.1237	20
50	.6670	.7451	.8952	1.1171	10
42° 0′	.6691	.7431	.9004	1.1106	**48° 0′**
10	.6713	.7412	.9057	1.1041	50
20	.6734	.7392	.9110	1 0977	40
30	.6756	.7373	.9163	1 0913	30
40	.6777	.7353	.9217	1 0850	20
50	.6799	.7333	.9271	1.0786	10
43° 0′	.6820	.7314	.9325	1.0724	**47° 0′**
10	.6841	.7294	.9380	1 0661	50
20	.6862	.7274	.9435	1 0599	40
30	.6884	.7254	9490	1 0538	30
40	.6905	.7234	.9545	1 0477	20
50	.6926	.7214	.9601	1.0416	10
44° 0′	.6947	.7193	.9657	1 0355	**46° 0′**
10	.6967	.7173	.9713	1 0295	50
20	.6988	.7153	.9770	1 0235	40
30	.7009	.7133	.9827	1 0176	30
40	.7030	.7112	.9884	1.0117	20
50	.7050	.7092	.9942	1.0058	10
45° 0′	.7071	.7071	1.0000	1.0000	**45° 0′**

9–8. Principal Value of an Angle. The smallest numerical value of an angle corresponding to a trigonometric function of given value is called the *principal value* of the angle. The principal value of $\sin^{-1}\left(\frac{1}{2}\right) = 30°$; the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is 60°; and the principal value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is ±45°. In the last case, where two values of opposite sign are given, the positive value alone is taken as the principal value of the angle.

An inverse trigonometric function is frequently understood to imply only the principal value of the angle.

Exercise 9–6

A. With reference to Table 9–1 find the principal values of:

1. $\sin^{-1} 0.3090$.
2. $\tan^{-1} 2.2286$.
3. $\cos^{-1} 0.3800$.

B. With reference to Fig. 9–5 and Table 9–1 find two values for each of the following (do not find the values of the corresponding angles):

1. $\cos \sin^{-1} 0.5592$.
2. $\tan \sin^{-1} 0.9580$.
3. $\sin \cos^{-1} 0.6820$.

9–9. Interpolation. Although Table 9–1 lists the trigonometric functions only for those angles which are integral multiples of 10′, it is possible to obtain functions of angles of intermediate value through an interpolation approximation. To illustrate the interpolation process let us find sin 34° 16′. From Table 9–1, sin 34° 10′ = 0.5616, and sin 34° 20′ = 0.5640. In a scale of values, 34° 16′ occupies a position six tenths of the way between 34° 10′ and 34° 20′. Then, if the sine of an angle increases uniformly with the angle, sin 34° 16′ should be in value six tenths of the way between sin 34° 10′ and sin 34° 20′. Strictly, the sine of an angle does not increase uniformly with the angle, but over a small interval of

angle, such as 10′, the approximation is quite good. In this way we obtain, then,

$$\begin{aligned}\sin 34^\circ 16' &= \sin 34^\circ 10' + [0.6 \cdot (\sin 34^\circ 20' - \sin 34^\circ 10')] \\ &= 0.5616 + 0.6\,(0.5640 - 0.5616) = 0.5630.\end{aligned}$$

For most practical communications work interpolation is unnecessary, and it is satisfactory to use simply the nearest figures which can be found in Table 9–1. On this basis $\sin 34^\circ 16' = 0.5640$.

Exercise 9–7

By interpolation in Table 9–1 find:

1. $\cos 76^\circ 27'$. 2. $\tan 12.3^\circ$. 3. $\sin^{-1} 0.4036$.

9–10. Defining Relations in Special Cases. In the definitive equations of the trigonometric functions, Eqs. (9–1), (9–2), and (9–3), the angle θ is considered such that its initial side is horizontal. In case we wish to

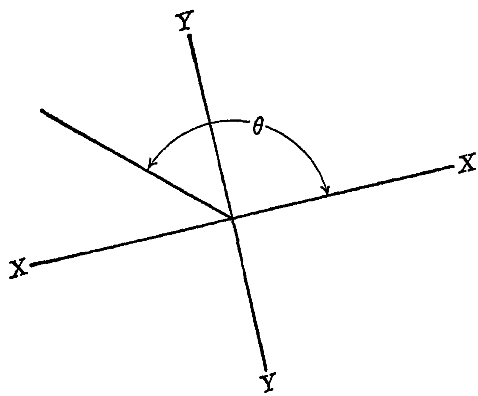


FIG. 9–6. X-axis coincident with initial side of θ.

consider an angle as in Fig. 9–6 whose initial side is not horizontal, we must then consider a set of coordinate axes rotated so that the x-axis coincides with the initial side of the angle.

It is frequently desirable to study relations among the sides and acute angles of a right triangle, without reference to coordinate axes. Under these circumstances we treat each angle and each side as positive regard-

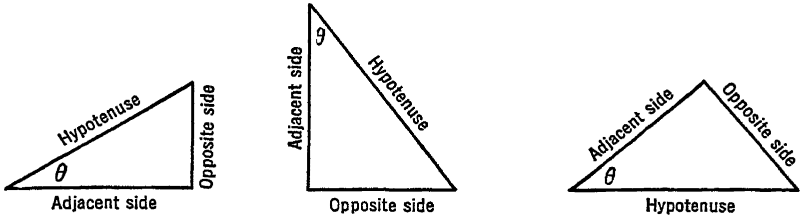


FIG. 9-7. Reference triangles for alternative definitions of the trigonometric functions of an acute angle.

less of the position of the triangle, and we define (Fig. 9-7):

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}; \tag{9-7}$$

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}; \tag{9-8}$$

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}. \tag{9-9}$$

Exercise 9-8

Show that for the particular case of acute angles the general definitions of Eqs. (9-1) through (9-3) are equivalent to the special definitions of Eqs. (9-7) through (9-9).

9-11. Relations among the Trigonometric Functions. By means of the definitive relations, Eqs. (9-1), (9-2), and (9-3), one can express any trigonometric function in terms of any other trigonometric function. Thus, from

$$\sin \theta = \frac{y}{r},$$

$$\cos \theta = \frac{x}{r},$$

and

$$x^2 + y^2 = r^2,$$

it follows that

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1. \tag{9-10}$$

Then from Eq. (9-10), $\sin^2\theta + \cos^2\theta = 1$, we have

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} \tag{9-11}$$

and

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}. \tag{9-12}$$

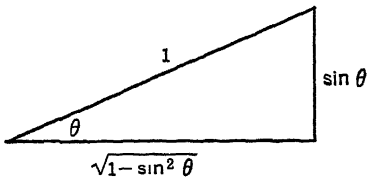


FIG. 9-8. Reference triangle for expressing any trigonometric function of θ in terms of $\sin\theta$.

It is sometimes expedient to obtain such a relationship as Eq. (9-11) or Eq. (9-12), which holds for all angles θ, from a consideration of the geometric relationships in a right triangle. If it is desired to express each of the functions, $\cos\theta$ and $\tan\theta$, in terms of $\sin\theta$, we construct a right triangle, as in Fig. 9-8, such that the hypotenuse is of unit length and the side opposite the angle θ is of length $\sin\theta$. Then the side adjacent to the angle θ is $\sqrt{1 - \sin^2\theta}$.

And we have at once

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \sqrt{1 - \sin^2\theta}; \tag{9-13}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}} \tag{9-14}$$

By this means we have expressed $\cos\theta$ and $\tan\theta$ in terms of $\sin\theta$ so that we can, if we so desire, obtain the values of $\cos\theta$ and of $\tan\theta$ directly from a given value of $\sin\theta$ without employing a trigonometric table or comparable means.

Eqs. (9-11) and (9-12) were obtained from the general definitive relations for $\sin\theta$ and $\cos\theta$ of Eqs. (9-1) and (9-2). Hence, Eqs. (9-11) and (9-12) are valid for all angles θ. On the other hand, Eqs. (9-13) and (9-14) were obtained from a consideration of the special case of a right triangle, and these equations are valid only for θ between 0° and 90°. However, the triangle scheme serves to evaluate absolute magnitudes of the relationships for any given angle. Thus, for any angle θ the magnitude of $\tan\theta$ is given by

$$|\tan\theta| = \left|\frac{\sin\theta}{\sqrt{1 - \sin^2\theta}}\right|;$$

Fig. 9-5 serves to indicate an appropriate sign for $\tan\theta$. By way of illustration, from Eq. (9-14) together with Fig. 9-5 we see that:

$$\text{for } \theta = 85^\circ \qquad \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}};$$

$$\text{for } \theta = 95^\circ \qquad \tan\theta = -\frac{\sin\theta}{\sqrt{1-\sin^2\theta}};$$

$$\text{for } \theta = -5^\circ \qquad \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

In many applications we shall be concerned with angles between 0° and 90° only, and in these cases reference to Fig. 9-5 is unnecessary.

If $\sin\theta$ is expressed as a ratio of two quantities, $\dfrac{A}{B}$, the auxiliary tri-

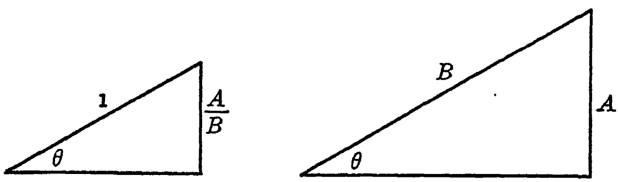


FIG. 9-9. Optional reference triangles for computing trigonometric functions of $\sin^{-1}\dfrac{A}{B}$.

angle may be modified so that the opposite side and hypotenuse are of lengths A and B, respectively, instead of $\dfrac{A}{B}$ and 1. Corresponding ratios of sides are the same for both triangles (Fig. 9-9).

Example. Given that θ is an acute angle and that $\sin\theta = 0.3$; find $\cos\theta$ and $\tan\theta$.

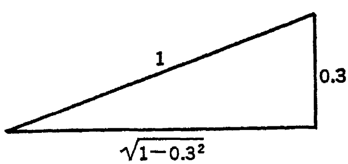


FIG. 9-10. Reference triangle for finding the trigonometric functions of $\sin^{-1} 0.3$.

A right triangle is constructed in Fig. 9-10 such that the hypotenuse is of unit length and the side opposite the angle θ is of length 0.3 unit. Then the

third side is of length $\sqrt{1 - 0.3^2}$, so that

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \sqrt{1 - 0.3^2}$$
$$= \sqrt{1 - 0.09} = \sqrt{0.91} = 0.954;$$

and $\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{0.3}{0.954} = 0.315.$

To express each of the trigonometric functions in terms of $\cos\theta$ we again construct a right triangle which is such that the hypotenuse is of unit length, but now we regard the side adjacent to the angle θ as of length $\cos\theta$. To express each of the functions in terms of $\tan\theta$ we construct a right triangle which is such that the side adjacent to the angle θ is of unit length and the side opposite the angle θ is of length $\tan\theta$.

Exercise 9–9

A. Solve the problem given in the example of Sec. 9–9 by using an auxiliary triangle of sides 3, 10, and $\sqrt{91}$, respectively, instead of 0.3, 1, and $\sqrt{0.91}$.

B. Construct appropriate auxiliary triangles, and develop expressions for:

1. $\sin\theta$ and $\tan\theta$ in terms of $\cos\theta$.
2. $\sin\theta$ and $\cos\theta$ in terms of $\tan\theta$.

C. Express each of the following as a function of u:

1. $\cos(\sin^{-1} u)$.
2. $\tan[\cos^{-1}(u + 2)]$.

D. Evaluate:

1. $\cos(\tan^{-1} 2)$.
2. $\cos\left(\tan^{-1}\dfrac{X}{R}\right)$.
3. $\tan(\cos^{-1}\frac{3}{5})$.
4. $\cos(\sin^{-1}\frac{5}{13})$.

E. Solve for x:

1. $\sin x + \cos x = 1$.
2. $\sin x = 3\cos x$.

9–12. Identities. An equation which is satisfied by every value of a variable is called an identity in that variable. Eqs. (9–11) and (9–12),

being satisfied by every value of θ, are identities in θ. The triple bar $\equiv$ may be employed to denote "is identically equal to." Thus,

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}. \tag{9–15}$$

Several useful identities are herewith presented without proof. These identities will be developed in Chapter 19.

$$\sin(A + B) = \sin A\cos B + \cos A\sin B \tag{9–16}$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B. \tag{9–17}$$

$$\sin A\sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)] \tag{9–18}$$

$$\cos A\cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{9–19}$$

Exercise 9–10

A. Prove Eq. (9–15).
B. Indicate the identities among the following equations:

1. $2x = 8x - 6x$.
2. $3y = 6$.
3. $\sin^2 x + \sin x - 1 = 0$.
4. $\cos^2\theta = \dfrac{1}{1 + \tan^2\theta}$.

9–13. Graphs of Functions Involving Sines and Cosines. Addition of Ordinates. Curves representing the functions $y = \sin\theta$, $y = 2\sin\theta$, and $y = \frac{1}{2}\sin\theta$ are plotted in Fig. 9–11. For each given value of θ the ordinate of the curve of $y = 2\sin\theta$ is twice the ordinate of the curve of $y = \sin\theta$, and the ordinate of the curve of $y = \frac{1}{2}\sin\theta$ is one half of the ordinate of the curve of $y = \sin\theta$.

Graphs of $y = \sin\theta$, $y = \sin 2\theta$, and $y = \sin\frac{1}{2}\theta$ are plotted in Fig. 9–12; and graphs of $y = \sin\theta$, $y = \sin(\theta + 30°)$, and $y = \sin(\theta - 30°)$ are plotted in Fig. 9–13.

For purposes of study it is frequently adequate to draw rough curves which indicate the general behavior of a function. Let us suppose it is desired to plot a graph of $y = 1 + 2\sin\theta$. One method of construction involves first sketching a graph of $y = 2\sin\theta$ and then adding 1 to each ordinate. This procedure is illustrated in Fig. 9–14(a). An alternative and preferred procedure is that illustrated in Fig. 9–14(b). Here the curve of $y = 2\sin\theta$ is sketched about the line $y = 1$ as an axis. In

either case it is evident from the figures that the ordinate at any point is given by $1 + 2 \sin \theta$ as required.

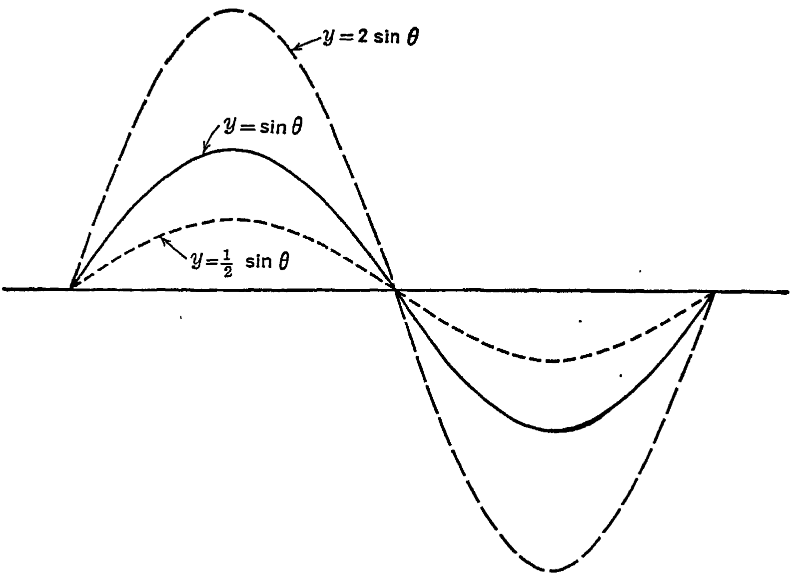


FIG. 9–11. Graphs of $y = \sin \theta$, $y = 2 \sin \theta$, and $y = \frac{1}{2} \sin \theta$.

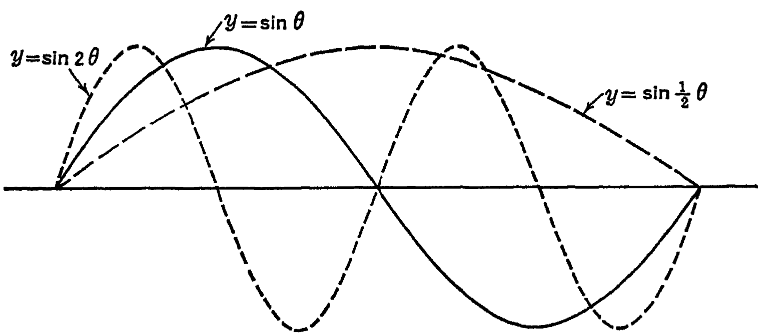


FIG. 9–12. Graphs of $y = \sin \theta$, $y = \sin 2\theta$, and $y = \sin \frac{1}{2}\theta$.

The graph of the function $y = 3 \cos \theta + \cos 2\theta$ is shown in Fig. 9–15. This graph is obtained by adding point-by-point the ordinates of the

curves $y = 3 \cos \theta$ and $y = \cos 2\theta$, which are shown dotted in Fig. 9–15. In the actual construction of such a graph for rough investigations the auxiliary curves, $y = 3 \cos \theta$ and $y = \cos 2\theta$, are first sketched approximately to scale, and then the final curve is obtained from them by adding ordinates only at certain points. The points selected are indicated by

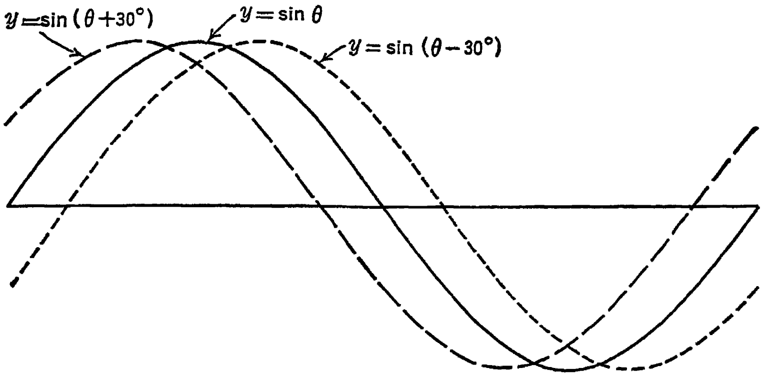


FIG. 9–13. Graphs of $y = \sin \theta$, $y = \sin(\theta - 30°)$, and $y = \sin(\theta - 30°)$.

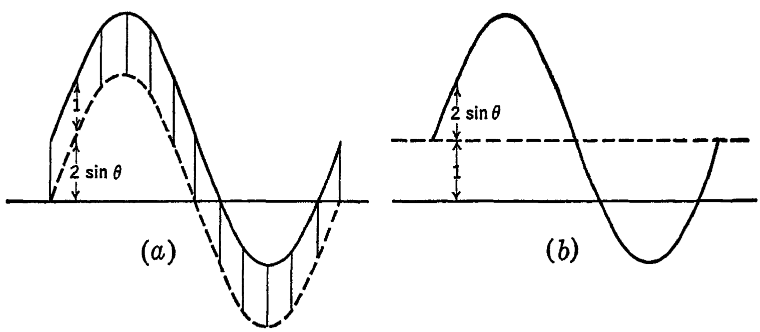


FIG. 9–14. Two methods for construction of the graph of $y = 1 + 2 \sin \theta$.

dots and labelled with letters in Fig. 9–15. Points B, D, F, and H correspond to zero values of $\cos 2\theta$. For the particular value of θ associated with each of these points the sum function, $3 \cos \theta + \cos 2\theta$, amounts to simply $3 \cos \theta$. C and G correspond to zero values of $3 \cos \theta$. And for the particular values of θ associated with each of these points the sum function, $3 \cos \theta + \cos 2\theta$, amounts to simply $\cos 2\theta$. Points A, E,

and I result from graphically adding 1 to each of the corresponding ordinates of the curve of 3 cos θ. At intervening points the nature of

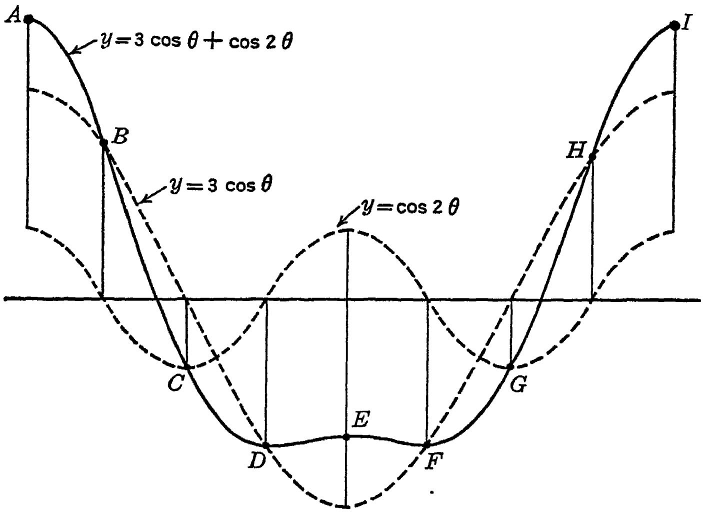


FIG. 9-15. Construction of graph of $y = 3 \cos \theta + \cos 2\theta$.

the curves of $y = 3 \cos \theta$ and of $y = \cos 2\theta$ suggests the course of the interpolation.

Exercise 9-11

A. Plot the following curves accurately, using the same set of coordinate axes for all three curves: $y = \sin \theta$, $y = \sin (\theta + 45°)$, $y = \sin (\theta - 15°)$. Plot points at intervals of 15° for θ from 0° to 360°.

B. Make rough plots (as separate graphs unless otherwise specified) of the following functions for values of θ between 0° and 360°.

1. $y = \frac{1}{2} \cos \theta$.
2. $y = -3 \sin \theta$.
3. $y = -\sin 2\theta$.
4. $y = 4 \cos 3\theta$.
5. $y = \sin \theta + \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta$.
6. $y = 2(\sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta)$.
7. $y = -10 + 2 \sin \theta$; $y = -5 + 2 \sin \theta$; $y = -1 + 2 \sin \theta$ (all on the same set of coordinate axes).
8. $y = \frac{1}{10}\theta + 2 \sin \theta$; $y = -5 + \frac{1}{10}\theta + 2 \sin \theta$ (both on the same set of coordinate axes).

9–14. The Reciprocal Functions. Cofunctions. The functions cosecant, secant, and cotangent are defined as the reciprocals of the functions sine, cosine, and tangent, respectively:

$$\frac{1}{\sin\theta} = \text{cosecant } \theta \text{ (abbreviated csc } \theta); \tag{9–20}$$

$$\frac{1}{\cos\theta} = \text{secant } \theta \text{ (abbreviated sec } \theta); \tag{9–21}$$

$$\frac{1}{\tan\theta} = \text{cotangent } \theta \text{ (abbreviated cot } \theta). \tag{9–22}$$

Values of the reciprocal functions are obtained by taking the reciprocals of the corresponding functions, thus:

$$\csc 30^\circ = \frac{1}{\sin 30^\circ} = \frac{1}{0.5} = 2.$$

The pairs of functions: sine-cosine, tangent-cotangent, secant-cosecant, are spoken of as *cofunctions.* It may be demonstrated (compare Problems 1 through 4 of Exercise 9–2) that any function of a given angle is equal to the cofunction of the complementary angle.

Exercise 9–12

A. Evaluate:

1. $\csc 45^\circ$.
2. $\sec(-30^\circ)$.
3. $\cot 12^\circ 10'$.
4. $\sec^{-1}\sqrt{2}$.
5. $\cot^{-1}(-1)$.
6. $\csc^{-1} 2.5$.

B. Prove the following identities:

1. $\dfrac{\csc\theta}{\sec\theta} = \cot\theta$.
2. $\sec^2 x = 1 + \tan^2 x$.
3. $\csc^2\alpha = 1 + \cot^2\alpha$.

C. Demonstrate:

1. That the tangent of any given angle is equal to the cotangent of the complementary angle.
2. That the secant of any given angle is equal to the cosecant of the complementary angle.

D. Reproduce Fig. 9–5, and superimpose on the same set of axes curves of the reciprocal functions. (Obtain values of the ordinates of the reciprocal curves from the ordinates of the sine, cosine, and tangent curves. Thus, at the abscissa for which $\sin\theta = \frac{1}{4}$, $\csc\theta = 4$; at the abscissa for which $\sin\theta = \frac{1}{2}$, $\csc\theta = 2$.)

9–15. Functions of Special Angles. Values of the trigonometric functions of 0°, 30°, 45°, 60°, and 90° are given in Table 9–2 for reference.

TABLE 9–2. FUNCTIONS OF SPECIAL ANGLES

Angle	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞

CHAPTER 10

RADIAN MEASURE OF ANGLES. AVERAGE VALUES

10–1. Radian Measure of Angles. Two systems of measurements of angles are used in scientific work. In one system angles are measured in the familiar unit, the degree. In the other system angles are measured in *radians*. Here we shall consider the radian.

In the diagram of Fig. 10–1, O is the center of three concentric circles. For the 30° angle shown, the ratio of the subtended arc to the corresponding radius is the same in each circle:

$$\frac{\overline{A_1B_1}}{R_1} = \frac{\overline{A_2B_2}}{R_2} = \frac{\overline{A_3B_3}}{R_3}. \qquad (10\text{–}1)$$

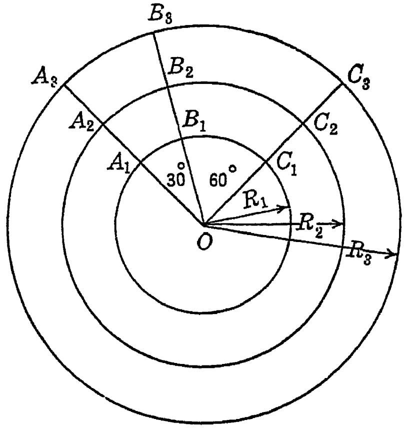


FIG. 10–1. Ratio of arc to radius proportional to the angle.

For the 60° angle the ratios of subtended arc to radius are likewise equal to each other:

$$\frac{\overline{B_1C_1}}{R_1} = \frac{\overline{B_2C_2}}{R_2} = \frac{\overline{B_3C_3}}{R_3}. \qquad (10\text{–}2)$$

Furthermore, any of the ratios in Eq. (10–2) is twice any of the ratios in Eq. (10–1). In general, for an angle formed at the center of a circle (*a central angle*) the ratio of arc to radius is directly proportional to the size of the angle and is independent of the length of the radius. This fact makes it possible to use as a measure of the size of an angle the ratio of arc to radius, where both lengths are measured in the same units. The unit angle in this system is the radian. The radian is a central angle which subtends an arc one radius in length (Fig. 10–2). In other words, the radian is a central angle which is measured by unit ratio of arc to radius:

$$\frac{\text{arc}}{\text{radius}} = 1.$$

The ratio of arc to radius for a central angle of 180° is π, so that in radians a 180° angle is given by π. Correspondingly, an angle of 90° is in radians equal to $\frac{90}{180} \cdot \pi = \frac{\pi}{2}$; and an angle of 30° is in radians equal to $\frac{30}{180} \cdot \pi = \frac{\pi}{6}$. An angle of x degrees is in radians equal to $\frac{x}{180} \cdot \pi$. Inasmuch as an angle of π radians is equal to 180°, an angle of one radian equals $\frac{180°}{\pi}$, or approximately 57°.

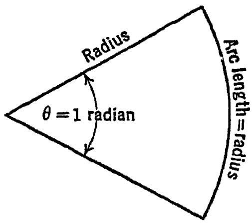


FIG. 10-2. Angle of 1 radian.

In scientific work either the radian or the degree is used, depending upon which is more convenient for the particular purpose. In operations involving calculus the radian is generally used because it makes for simplicity of the formulas. (See Sec. 11-11.)

Whenever an angle is specified simply by a number, that is, without an accompanying unit, it is understood that the unit is the radian. The statement "$\theta = \pi$" implies "$\theta = \pi$ radians." For an angle in degrees the unit, degree, is always explicitly included with the numerical value; thus, 180°.

Exercise 10-1

A. Convert the following angles from radians to degrees:

1. π radians. **2.** $\frac{3}{2}\pi$ radians.

B. Convert the following angles from degrees to radians:

1. 90°. **3.** 135°.
2. 45°. **4.** 17°.

C. A wheel rotates at the rate of 60 revolutions per second. How many radians per second is this?

D. What length of arc is subtended (a) by a central angle of 2 radians in a circle of radius 3 units? (b) By a central angle of x radians in a circle of radius 1 unit?

E. Using the radian as the unit of angle, plot rough graphs of the following functions for θ from 0 to 2π:

1. $y = \sin \theta$. Designate points on the x-axis corresponding to $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{3}, \pi, \frac{3}{2}\pi$, and 2π.

2. $y = \theta$. (This is the same as $y = x$ where x is measured in radians.)

3. $y = -\theta$.

4. $y = \theta \sin \theta$. (First plot on the same set of axes the auxiliary curves $y = \theta$, $y = -\theta$, and $y = \sin \theta$. For any abscissa θ, the corresponding ordinate is the product of θ and $\sin \theta$.)

5. $y = \frac{1}{2}x + \cos x$.

F. In the following functions t denotes time in seconds and ω is a constant with dimensions of radians per second, so that ωt is an angle in radians. Plot accurately curves of each of the following functions, indicating points at intervals of 0.01 second from $t = 0$ second to $t = 0.1$ second.

1. $y = \sin(\omega t)$ and $y = \sin\left(\omega t + \dfrac{\pi}{10}\right)$ where $\omega = 20$ radians per second. (Plot both curves on the same set of coordinate axes.)

2. $y = 2 \cos(\omega t)$ where $\omega = 10$ radians per second.

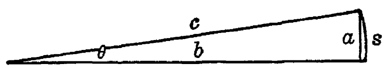


FIG. 10–3. $\theta \approx \sin\theta \approx \tan\theta$.

10–2. Approximations. We observe from Fig. 10–3, wherein θ is a small angle (less than 10°), that

$$\sin \theta = \frac{a}{c} \approx \frac{a}{b} = \tan \theta.$$

For θ measured in radians, that is, for θ measured by the ratio of arc to radius, $\dfrac{s}{c}$, we observe further that

$$\sin \theta \approx \theta \approx \tan \theta. \qquad (10\text{–}3)$$

The approximations of Eq. (10–3) facilitate certain computations involving small angles.

Exercise 10–2

Functions of 1°, 5°, and 10° are given herewith:

Angle	Sin	Tan
1°	0.01745	0.01746
5°	0.08716	0.08749
10°	0.1736	0.1763

Find the percentage error in each of the following approximations: (a) for $\theta = 1°$, (b) for $\theta = 5°$, and (c) for $\theta = 10°$ (in each case consider that the expression on the right side represents the correct value):

1. $\sin\theta \approx \theta$ (θ in radians).
2. $\tan\theta \approx \theta$ (θ in radians).
3. $\sin\theta \approx \tan\theta$ (θ in radians).

10–3. Average Value of a Function. In Fig. 10–4 is shown a curve representing the general relation $y = f(x)$. Assuming an intuitive understanding of the concept of "area," we define the *average value of* $f(x)$ *between abscissas corresponding to* B *and* C as the area $ABCD$ divided by the length of the base AD. In other words, the average value of $f(x)$ between B and C is y_0, the altitude of a rectangle $AB'C'D$ whose area is equal to $ABCD$, the area under the curve.

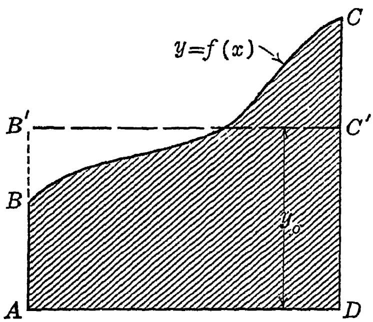


Fig. 10–4. Area $AB'C'D$ = area $ABCD$. y_0 = average value of $f(x)$.

The word "area" is here used in a broad sense. If we express x in inches and y in inches, then the area is in square inches, and the average ordinate is in units of square inches divided by inches, or in inches. If we express x in radians and y in volts, then the area is in units of radians times volts; and the average ordinate is in units of radians times volts divided by radians, or in volts. For our present purposes we shall regard an area which lies above the x-axis as *positive* and an area which lies below the x-axis as *negative*. With this convention the net area between the curve of $y = \sin\theta$ and the x-axis from $\theta = 0°$ to $\theta = 360°$ is zero; and, hence, the average ordinate of the sine curve over this interval is zero.

In this chapter we shall evaluate certain averages which are fundamental in alternating current theory.

Exercise 10–3

1. Evaluate the average ordinate y_0 for each of the curves of Fig. 10–5.
2. Show that the average value of $\sin\theta$ is zero:

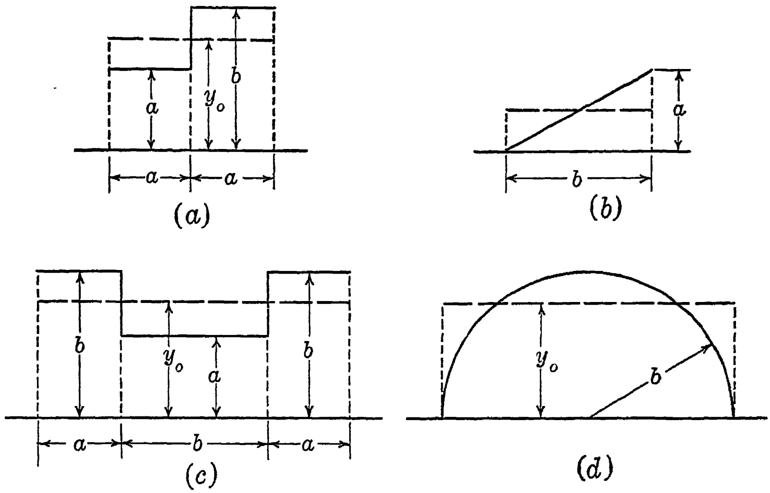


FIG. 10-5. Miscellaneous simple curves; average ordinates y_0.

a. over the interval from $\theta = \frac{\pi}{2}$ to $\theta = \frac{5}{2}$ radians;

b. over the interval from $\theta = -30°$ to $\theta = 330°$;

c. over any interval which is an integral multiple of 2π radians.

3. Find the average value of:

a. $y = 2 + \sin \theta$;

b. $y = 3 - 4 \sin (\theta + 15°)$;

c. $y = A + B \sin (\theta + C)$, where A, B, and C are constants;

d. $y = A + B \cos (\theta + C)$;

e. $y = B \cos (A\theta + C)$.

10-4. Average Value of Sin² θ and of Cos² θ. RMS. Finding the average value of $\sin^2 \theta$ is simplified by the fact that the curve of $y = \sin^2 \theta$ [Fig. 10-6(a)] is symmetrical in many respects. A detailed study of Fig. 10-6(a) would show that: (1) all loops of the curve of $y = \sin^2 \theta$ are identical in size and shape; (2) for any one loop the right half, R, is a mirror image of the left half, L; (3) the region of a half-loop, such as L, is identical in size and shape with the associated adjacent region, T, which lies between the curve and the line $y = 1$.

To illustrate the manner of demonstrating these symmetries let us here prove just the symmetry of the third type. This requires us to show

that in the magnified half-loop of Fig. 10-6(b) any point P which is θ to the right of, and c above A, has its counterpart in a point Q which is

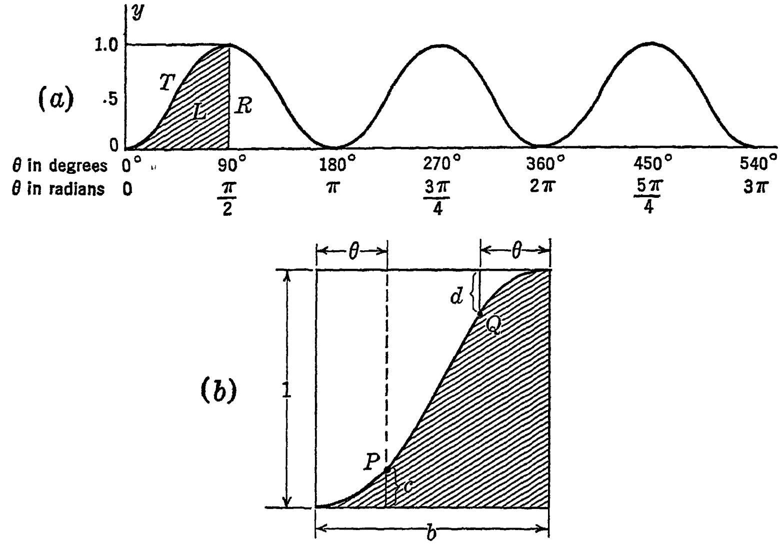


FIG. 10-6. (a) Curve of $y = \sin^2 \theta$;
(b) half loop (to magnified scale).

θ to the left of, and d below B. In other words, we need to prove that $d = c$. The proof is as follows:

$$\begin{aligned} d &= 1 - \sin^2 (90 - \theta) \\ &= 1 - \cos^2 \theta \\ &= \sin^2 \theta = c. \end{aligned}$$

Now according to the definition of the average value of a function, to obtain the average value of $\sin^2 \theta$ over the interval shown in Fig. 10-6(a), that is, from $\theta = 0$ to $\theta = 3\pi$, we must divide the total area under the three loops by the length of the base. But by the symmetrical properties of the curve of $y = \sin^2 \theta$ this average is the same as that over any integral number of half-loops. For simplicity, then, let us evaluate the average value of $\sin^2 \theta$ over a single half-loop.

The rectangular area $T + L$ of Fig. 10-6(a) equals the product of the base and the altitude, or $b \cdot 1$, where b denotes the length of the base and where the altitude is 1. Since the areas T and L are equal to each other,

each is equal to $\frac{b \cdot 1}{2}$. The average value of $\sin^2 \theta$ over a half-loop is then $\frac{\text{area}}{\text{base}} = \frac{b/2}{b} = \frac{1}{2}$. And this, as noted in the preceding paragraph, is also the average value of $\sin^2 \theta$ over any integral number of half-loops.

The square root of this average value of $\sin^2 \theta$, $\sqrt{\frac{1}{2}}$, is called the *root mean square* — abbreviated rms — value of sin θ. *Root* here denotes "square root," and mean denotes "average." The root mean square value of a function is the square root of the average square value of the function.

In the foregoing we have found:

$$\text{av. } \sin^2 \theta = \tfrac{1}{2},$$

from which:

$$\text{rms value of } \sin \theta = \sqrt{\text{av. } \sin^2 \theta} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} = 0.707. \qquad (10\text{-}3)$$

The curve of $y = \cos^2 \theta$ is identical with that of $y = \sin^2 \theta$ except for a displacement along the x-axis. Hence, the rms value of cos θ is likewise $\sqrt{\frac{1}{2}}$, or 0.707.

Exercise 10–4

1. For what angles does the instantaneous value of sin θ equal the rms value?
2. For what angles does the instantaneous value of cos θ equal the rms value?
3. Find the rms value of:
   a. $2 \sin \theta$;
   b. $2 \sin (\theta - 15°)$;
   c. $I_m \sin \theta$;
   d. $I_m \sin (\theta - \beta)$ where β is a constant.
4. Compute the rms value of the square wave of maximum value E_m shown in Fig. 10–7.

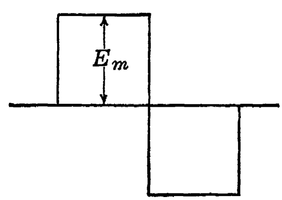


FIG. 10–7. Square Wave.

10–5. Half-Wave Average Value of Sin θ and of Cos θ. A frequently occurring quantity in electrical work is the average value of sin θ or of cos θ over any positive loop of the corresponding curve. Since the two sections of the sine or cosine curve represented in Fig. 10–8(a) are symmetrical with respect to the line AB, the average value of the function

over the complete interval of Fig. 10-8(a) is the same as the average over either part.

Let us consider the right portion of the figure. The curve of Fig. 10-8(b) is that of $\cos\theta$ over the interval of θ from 0 to $\frac{\pi}{2}$ radians. The area under this curve is given approximately by the sum of the areas of

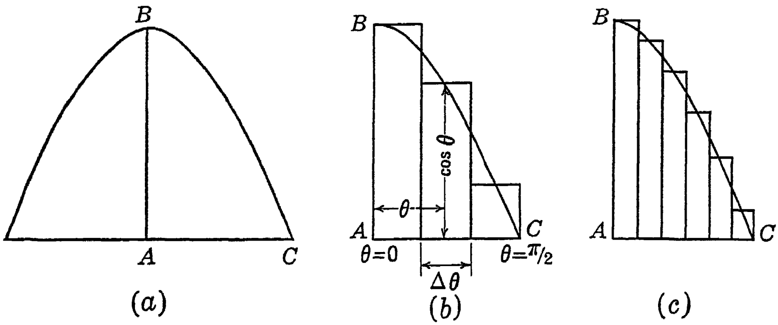


FIG. 10-8. Method for determination of the area under a portion of the cosine curve.

the three rectangles shown. For any one rectangle the height is given by the value of $\cos\theta$ at the center of the rectangle, and the width is a small angle, which we might call $\Delta\theta$*. For simplicity we consider all of the rectangles of the same width $\Delta\theta$. If we designate the rectangles by numbers, and if we express $\Delta\theta$ in radians, the area of the first rectangle is $\Delta\theta \cos\theta_1$, of the second, $\Delta\theta\cos\theta_2$, and of the third, $\Delta\theta\cos\theta_3$.

Now with reference to (c) of Fig. 10-8 we see that the desired area ABC is best approximated by the sum of the elementary rectangles when $\Delta\theta$, the width of each rectangle, is taken as small as possible. Strictly, then, what we aim to evaluate is the limit which is approached by the sum S_n of the rectangular areas,

$$S_n = \Delta\theta\cos\theta_1 + \Delta\theta\cos\theta_2 + \cdots + \Delta\theta\cos\theta_n, \tag{10-4}$$

as $\Delta\theta$ approaches zero (or, what amounts to the same thing, as n, the number of rectangles increases indefinitely).

To assist us in the evaluation of the limit of S_n we construct the auxiliary diagrams of Fig. 10-9 in which the curve shown is the arc of a circle

* $\Delta\theta$ is read: "delta theta." $\Delta\theta$ is a single entity, namely, a small angle, and it does not imply the product of two terms, Δ and θ.

of radius 1. It is to be emphasized that the curve of Fig. 10–9 is not itself a section of a cosine curve, but that it is the arc of a circle whose function is to assist our determination of the area ABC under the portion of the cosine curve which is shown in Fig. 10–8. By laying out angles, each $\Delta\theta$ in magnitude, we divide the arc in Fig. 10–9(a) into intervals,

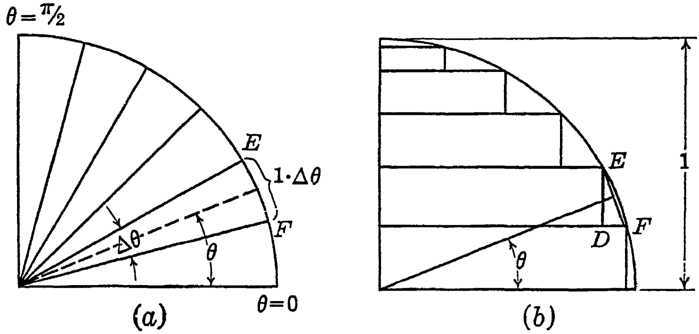


FIG. 10–9. Auxiliary construction for evaluating the area under a portion of the cosine curve.

each of length $1 \cdot \Delta\theta$ (compare Problem D of Exercise 10–1). Through the end point of each such interval we draw the horizontal lines shown in Fig. 10–9(b). In the triangle DEF the side DE is vertical, and the side EF is perpendicular to the radius which is shown at the angle θ. Hence, the angle DEF is equal to θ. Then, if we designate the length of the side EF by $\overline{EF}$, and the length of the arc EF by $\widehat{EF}$, we have

$$\frac{\overline{DE}}{\overline{EF}} = \cos\theta,$$

or

$$\overline{DE} = \overline{EF}\cos\theta. \tag{10–5}$$

But since

$$\overline{EF} \approx \widehat{EF} = \Delta\theta, \tag{10–6}$$

Eq. (10–5) becomes

$$\overline{DE} \approx \Delta\theta\cos\theta. \tag{10–7}$$

Since the approximation of Eq. (10–6) approaches an exact equation as $\Delta\theta$ approaches zero, the approximation Eq. (10–7) likewise approaches an exact equation. This last result is significant. It indicates that any

one elementary area, $\Delta\theta \cos\theta$, in Fig. 10-8 is measured approximately by the length $\overline{DE}$ in Fig. 10-9(b); and that as $\Delta\theta$ approaches zero the approximation is improved. Thus, the sum of the vertical lengths $\overline{DE}$, as θ extends from 0 to $\frac{\pi}{2}$, is the limit of the sum S_n of Eq. (10-4); in other words, it is the area ABC. The sum of the vertical lengths is equal to 1, the radius of the circle. Hence, the area under the curve ABC of Fig. 10-8 is 1; and the average value of the cosine function, or of the sine function, over the interval from $\theta = 0$ to $\theta = \frac{\pi}{2}$ is

$$\text{av. } \cos\theta = \text{av. } \sin\theta = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} = 0.637. \tag{10-8}$$

Exercise 10-5

1. Find the half-wave average value of the isosceles trianglar wave of maximum value E_m shown in Fig. 10-10.

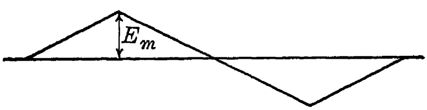


FIG. 10-10. Isosceles triangular wave.

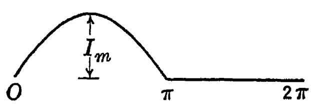


FIG. 10-11. Ideal rectifier curve.

2. Find the half-wave average value of

a. $2 \sin\theta$;

b. $2 \sin(\theta - 15°)$;

c. $I_m \sin\theta$;

d. $I_m \sin(\theta - \beta)$ where β is a constant.

3. Find the average value from $\theta = 0$ to $\theta = 2\pi$ for the function of maximum value I_m which is represented in Fig. 10-11. The curve is a sine curve for which the negative portion has been replaced by a segment of the x-axis.

10-6. Form Factor. The *form factor* of a wave is the ratio of the rms value to the average value. Form factors of typical waves are listed in Table 10-1.

TABLE 10–1. FORM FACTORS OF TYPICAL WAVES

Wave	Form Factor
Sine	$\frac{\pi}{2\sqrt{2}} = 1.11$
Half-wave rectified sine	$\frac{\pi}{2} = 1.57$
Full-wave rectified sine	$\frac{\pi}{2\sqrt{2}} = 1.11$
Rectangular	1
Half-wave rectified rectangular	$\sqrt{2} = 1.41$
Isosceles triangular	$\frac{2}{\sqrt{3}} = 1.15$
Half-wave rectified isosceles triangular	$\frac{4}{\sqrt{6}} = 1.63$

Exercise 10–6

1. Verify each of the first five form factors listed in Table 10–1.
2. Show that for *any* given type of wave: (a) the form factor of the full-wave rectified wave is the same as that for the given wave; (b) the form factor of the half-wave rectified wave is $\sqrt{2}$ times that for the given wave.

10–7. Average Power. If

$$e = E_m \sin \theta$$

is the electromotive force applied to a circuit, and if

$$i = I_m \sin (\theta + \beta),$$

where β is a constant, is the current flowing in the circuit, then the instantaneous power in the circuit is given by

$$\begin{aligned} p &= ei \\ &= E_m \sin \theta \cdot I_m \sin (\theta + \beta) \\ &= E_m I_m \sin \theta \sin (\theta + \beta). \end{aligned} \tag{10-9}$$

Let us recall the identity

$$\sin A \sin B = \tfrac{1}{2}[\cos (A - B) - \cos (A + B)] \tag{9-18}$$

which was stated in Sec. 9–12. On treating θ in Eq. (10–9) as analogous to A in Eq. (9–18), and $(\theta + \beta)$ in Eq. (10–9) as analogous to B in Eq. (9–18), we can expand Eq. (10–9) with the aid of Eq. (9–18) to obtain

$$p = \frac{E_m I_m}{2} [\cos(-\beta) - \cos(2\theta + \beta)]$$

$$= \frac{E_m I_m}{2} \cos\beta - \frac{E_m I_m}{2} \cos(2\theta + \beta). \tag{10–10}$$

Introducing E as the rms value of the electromotive force and I as the rms value of the current, we can write Eq. (10–10) as

$$p = EI \cos\beta - EI \cos(2\theta + \beta) \tag{10–11}$$

on replacing $\frac{E_m I_m}{2}$ by its equal EI. From Eq. (10–11), where the term $EI \cos\beta$ is a constant and where the term $EI \cos(2\theta + \beta)$ is of zero average value, the average power is seen at once to be $EI \cos\beta$. (Refer to Problem 3e of Exercise 10–3.)

Exercise 10–7

1. Find the average power in a circuit for an applied electromotive force $e = E_m \cos(\theta + \beta)$ and a current $i = I_m \cos\theta$.

2. Find the value of the average power for the square waves of current and voltage shown in Fig. 10–12(a), (b), and (c). Take maximum current as I_m, maximum voltage as E_m.

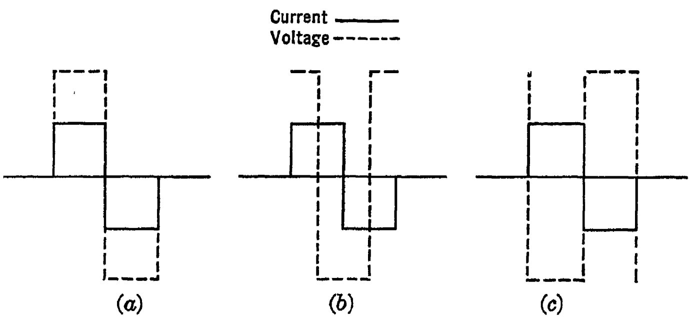


FIG. 10–12. Square waves of current and voltage.

CHAPTER 11

RATE OF CHANGE

11–1. Directed Line Segments. A line segment which joins points A and B, as in Fig. 11–1, is conventionally regarded as directed from A to B or from B to A in accordance with whether we write AB or BA, respectively. One direction is regarded as positive, and the opposite direction is regarded as negative. Unless otherwise specified the positive direction of all lines parallel to the x-axis is to the right; on all other lines the positive direction is upward. In Fig. 11–1 the segment lengths are related, then, through the equations:

FIG. 11–1. Line segment.

$$\overline{AC} = -\overline{CA}; \tag{11–1}$$

$$\overline{CB} = -\overline{BC}; \tag{11–2}$$

$$\overline{AB} = -\overline{BA}; \tag{11–3}$$

$$\overline{AC} + \overline{CB} = \overline{AB}. \tag{11–4}$$

Exercise 11–1

A. Show with reference to Fig. 11–1 [but without reference to Eqs. (11–1) through (11–4)] that:

1. $\overline{AC} - \overline{BC} = \overline{AB}$.
2. $\overline{AC} + \overline{BA} = \overline{BC}$.
3. $\overline{CA} + \overline{BC} = \overline{BA}$.
4. $\overline{BC} - \overline{AC} = \overline{BA}$.

B. Show that each of the relations in part A of this exercise may be obtained directly from Eqs. (11–1) through (11–4) without further reference to Fig. 11–1.

11–2. Projections of a Directed Line Segment on the Coordinate Axes. For a line segment joining points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ as in Fig. 11–2, we can show that the length of the projection on the x-axis is given by $x_2 - x_1$, and the length of the projection on the y-axis is given by $y_2 - y_1$. To prove this we construct the following additional lines in Fig. 11–2:

P_1M and NP_2 parallel to the x-axis, and P_1A and BP_2 parallel to the y-axis. Then, the length of the x-projection of the segment P_1P_2 is

$$\overline{AB} = \overline{AO} + \overline{OB} = \overline{P_1M} + \overline{NP_2} = \overline{NP_2} - \overline{MP_1}.$$

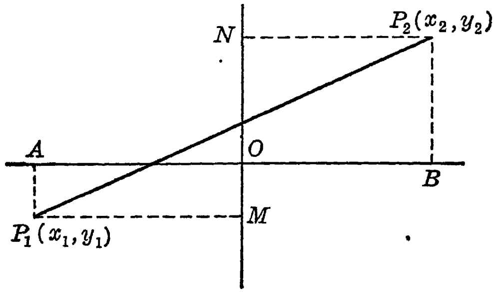


FIG. 11-2. x- and y-projections of line segment P_1P_2.

But x_2 is the distance from N to P_2, or $\overline{NP_2}$; and x_1 is the distance from M to P_1, or $\overline{MP_1}$. Thus,

$$\overline{AB} = x_2 - x_1.$$

The length of the y-projection of the segment P_1P_2 is

$$\overline{MN} = \overline{MO} + \overline{ON} = \overline{P_1A} + \overline{BP_2} = \overline{BP_2} - \overline{AP_1}.$$

But y_2 is the distance from B to P_2, or $\overline{BP_2}$; and y_1 is the distance from A to P_1, or $\overline{AP_1}$. Thus,

$$\overline{MN} = y_2 - y_1.$$

Since the x-axis is usually drawn horizontally, and the y-axis vertically, the projections are often referred to as the *horizontal* and *vertical projections*.

Exercise 11-2

1. Show that the lengths of the projections of P_2P_1 in Fig. 11-2 are $x_1 - x_2$ and $y_1 - y_2$.

2. Show that the lengths of the projections of P_1P_2 are $x_2 - x_1$ and $y_2 - y_1$:

a. when P_1 is in the first quadrant and P_2 is in the second quadrant;

b. when P_1 is in the fourth quadrant and P_2 is in the second quadrant.

11-3. Slope of a Line. The *slope of a line* is the ratio of the lengths of the vertical and horizontal projections of any segment of the line. It is apparent from a consideration of the line LL' in Fig. 11-3 that the ratio

of the lengths of the vertical and horizontal projections is the same for every segment of a given line. Hence, it is adequate to determine the slope of a line by means of any convenient segment.

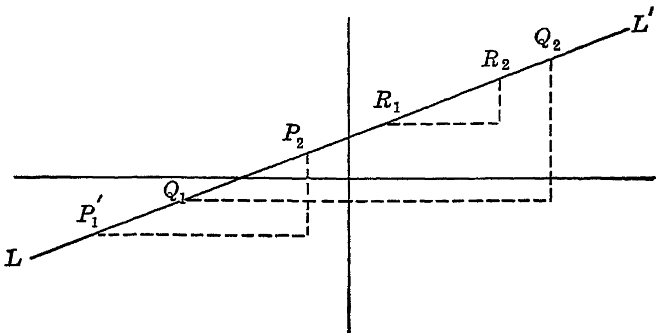


FIG. 11-3. Optional segments for determination of slope of line.

If, in Fig. 11-3, the coordinates of Q_1 are $(-8,-1)$ and the coordinates of Q_2 are $(9,6)$, then the slope of LL' as determined from the segment Q_1Q_2 is

$$m = \frac{6-(-1)}{9-(-8)} = \frac{7}{17} = 0.41.$$

The slope of LL' as determined from the segment Q_2Q_1 is (compare Problem 1 of Exercise 11-2)

$$m = \frac{-1-6}{-8-9} = \frac{-7}{-17} = 0.41,$$

the same as obtained above. These relations illustrate the fact that the slope of a line is independent not only of the particular segment, but also of the *sense* of direction of that segment which is chosen for the determination of the slope, wherein by " sense " we distinguish between the two possible directions along the line. The argument in the specific illustration above can readily be generalized to apply to any line. For a line joining points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, the slope as determined from P_1P_2 is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \tag{11-5}$$

and, as determined from P_2P_1, is given by

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We conclude that the selection of a segment of a line for the determination of the slope of that line is wholly arbitrary.

It follows from the definition of slope that if two lines are parallel their slopes are equal; and if two lines have the same slope they are parallel.

11–4. Sign of the Slope of a Line. It is a consequence of the definition of slope that lines which are directed up to the right or down to the left are of positive slope; and lines which are directed up to the left and down to the right are of negative slope. Lines parallel to the x-axis are of zero slope, and lines which are parallel to the y-axis are of infinite slope.

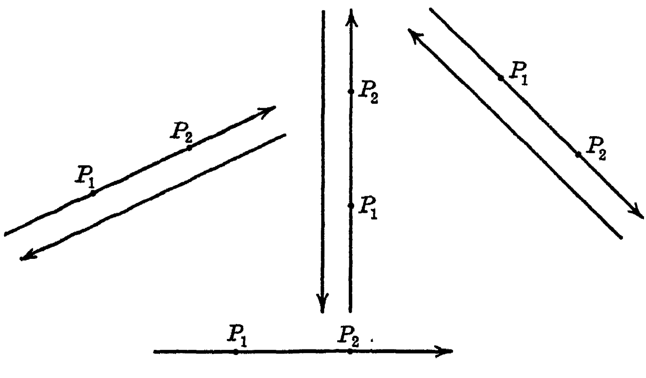


FIG. 11–4. Directed lines.

To prove the foregoing statements we might examine in turn each of the lines whose directions are indicated by arrows in Fig. 11–4. Since, however, for any particular line the slope as determined from the projections of a segment is independent of the sense of direction of the segment, it follows that of the four pairs of parallel lines in Fig. 11–4 we need consider only one line in each pair. In any case, for a segment P_1P_2 where P_1 has coordinates (x_1,y_1) and P_2 has coordinates (x_2,y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

For the segment P_1P_2 of the up-right line

$$y_2 > y_1, \text{ and } x_2 > x_1,$$

and, hence, m is positive. For the segment P_1P_2 of the down-right line

$$y_2 < y_1, \text{ and } x_2 > x_1,$$

so that m is negative. For the segment P_1P_2 of the horizontally-right line

$$y_2 = y_1;$$

and, consequently,

$$y_2 - y_1 = 0,$$

and m is zero. For the segment P_1P_2 of the vertically-up line

$$x_2 = x;$$

and, consequently,

$$x_2 - x_1 = 0,$$

and m is infinite.

Exercise 11-3

Draw lines through each of the following pairs of points. Determine the slope in each case.

1. (1,1), (6,6).
2. (0,0), (4,8).
3. (3,2), (−4,2).
4. (−3,5), (8,2).
5. (4,−3), (−2,−3).
6. (−4,4), (2,−6).
7. (5,−2), (−3,−8).
8. (−5,7), (−5,−2).

11-5. Scale Units and Slope. For any given linear equation the slope of the corresponding plotted line is independent of the choice of ordinate and abscissa scale units. The graph of $y = 4x$ is plotted four times in Fig. 11-5. In (a) the scale units of abscissas and ordinates are the same. In (b), (c), and (d) the scale of ordinates is successively compressed so that the line rotates clockwise in passing from (a) to (d). In each case, however, for the segment indicated in the figure, the ratio of the measured vertical projection to the measured horizontal projection is the same, namely, 4.

The *angle of inclination* of a line is the angle α through which the x-axis must be rotated to become parallel to (or coincident with), and in the same sense of direction as, the line. In determining the slope of a

line from a given graph of the line, it is sometimes convenient to evaluate $\tan \alpha$ rather than m. This is because $\tan \alpha$ may be determined through direct measurements on horizontal and vertical projections of a line segment with an inch rule; m is then obtained from $\tan \alpha$ by multiplying

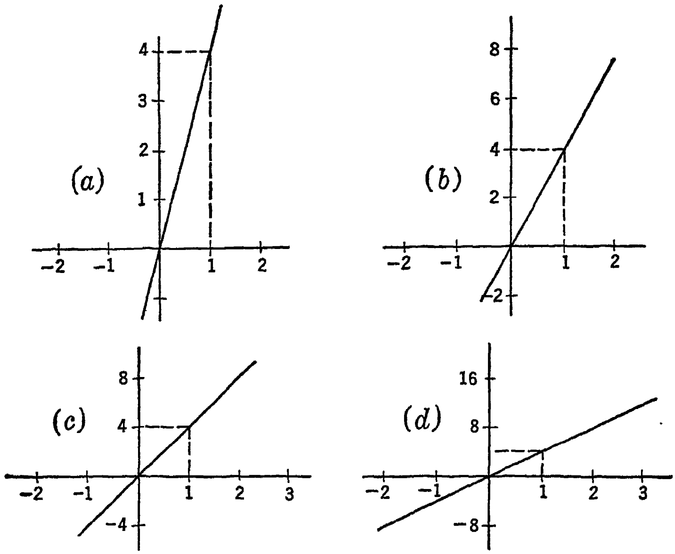


FIG. 11-5. Graph of $y = 4x$ showing the effect of various choices of scale units.

by an appropriate conversion factor. For any graph in rectangular coordinates let us designate as a *scale factor*, f, the number of ordinate units which occupy the same space on the graph as one abscissa unit. In (a) of Fig. 11-5, $f = 1$; in (b) $f = 2$; in (c) $f = 4$; and in (d) $f = 8$. In any case, we have for the relation between slope and the tangent of the angle of inclination

$$m = f \tan \alpha.* \tag{11-6}$$

The use of a scale factor correction is, of course, unnecessary if one

* In pure mathematics, where the scale of units is the same for both ordinates and abscissas, that is, where $f = 1$, the slope of a line is defined as equal to $\tan \alpha$. Frequently, in linear graphs of Ohm's Law relationships (for example, in so-called load lines) $\tan \alpha$ is described as conductance, or as inverse resistance, with dimensions of amperes times reciprocal volts. The slope of the line, not $\tan \alpha$, is implied here. Tan α, as any trigonometric function, is a dimensionless quantity.

always measures horizontal and vertical lengths on a graph in terms of those same horizontal and vertical units in which the graph is plotted.

Example. A line representing the current-voltage relationship for a certain resistor is shown in Fig. 11-6. Determine the slope of this line.

First Method. Any convenient segment of the line is adequate for the determination of the slope. We arbitrarily select a segment which extends from the origin to a point P of abscissa 2.00 volts. The ordinate of P as read from the scale of ordinates is 220 milliamperes. Then, by Eq. (11-5)

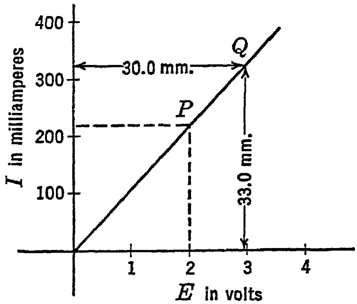


FIG. 11-6. Graph of current versus voltage for resistor. Determination of slope of line.

$$m = \frac{(220-0)\text{ milliamperes}}{(2.00-0)\text{ volts}}$$
$$= 110 \text{ milliamperes per volt.}$$

Second Method. We arbitrarily select a segment which extends from the origin to a point Q, where Q is such that its distance from the y-axis, as measured by a rule placed on the graph, is 30.0 millimeters. The height of Q above the x-axis is measured to be 33.0 millimeters. Then,

$$\tan \alpha = \frac{33.0}{30.0} = 1.10.$$

The scale factor is 100 milliamperes per volt. Hence, by Eq. (11-6),

$$m = 1.10 \cdot 100 \text{ milliamperes per volt}$$
$$= 110 \text{ milliamperes per volt.}$$

A millimeter rule is used in preference to an inch rule because the millimeter scale is in convenient decimal units whereas the inch scale is in cumbersome sixteenths.

Exercise 11-4

1. A graph representing the characteristics of an ideal full-wave rectifier is shown in Fig. 11-7. Determine the slope of each portion of the graph.

2. The characteristics of one type of wind-operated generator for charging batteries is shown in Fig. 11-8. Determine the slope of this graph for the region of wind velocity below 20 miles per hour and for the region of wind velocity above 20 miles per hour.

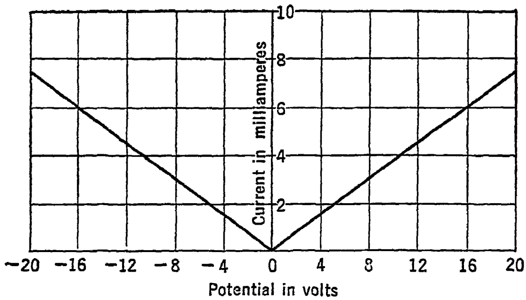


FIG. 11-7. Characteristics of an ideal full wave rectifier.

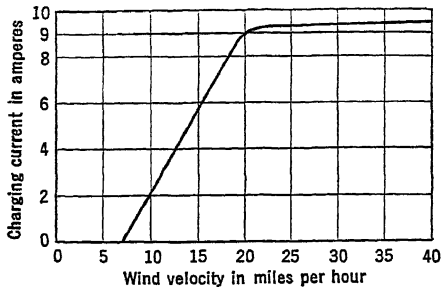


FIG. 11-8. Characteristics of a wind charger.

11-6. Slope of a Curve. By definition the *slope of a curve at a point* is the slope of the tangent line to the curve at that point.

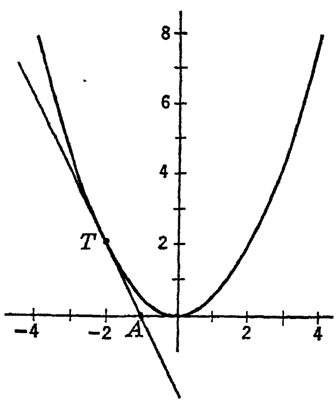


FIG. 11-9. Determination of the slope of the curve $y = \frac{1}{2}x^2$ at $x = -2$.

Example 1. Determine the slope of the curve $y = \frac{1}{2}x^2$ at the point corresponding to $x = -2$. The graph of $y = \frac{1}{2}x^2$, which was previously plotted in Fig. 3-6, is reproduced here in Fig. 11-9. We first construct the tangent line to the curve at the point corresponding to $x = -2$. A segment of this tangent line which happens to be convenient for the determination of its slope is the segment between T, the point where the line is tangent to the curve, and A, the point where the line intersects the x-axis. As read from the graph the coordinates of T are $(-2,2)$, and the coordinates of A are $(-1,0)$. Then, by Eq. (11-5) the

slope of the tangent line is

$$m = \frac{(0 - 2)}{-1 - (-2)} = -2.$$

And this, by definition, is the slope of the curve $y = \frac{1}{2}x^2$ at the point corresponding to $x = -2$.

Example 2. The plot of a sinusoidal emf curve $e = 10 \sin x$ volts is shown in Fig. 11–10. Determine the slope of this curve at the point corresponding to $x = \frac{\pi}{4}$ radian.

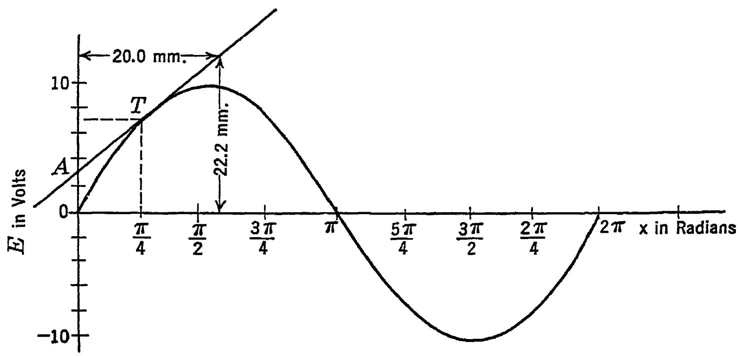


FIG. 11–10. Determination of the slope of the curve $e = 10 \sin x$ volts at $x = \frac{\pi}{4}$.

First Method. We arbitrarily select a segment of the tangent line which extends from A, the point of intersection with the y-axis, to T, the point of tangency to curve. The ordinate of A as read from the scale of ordinates is 1.5. The ordinate of T as read from the scale of ordinates is 7.1. Then, by Eq. (11–5)

$$m = \frac{(7.1 - 1.5) \text{ volts}}{\left(\frac{\pi}{4} - 0\right) \text{radian}}$$
$$= 7.1 \text{ volts per radian.}$$

The slope of this curve at $x = \frac{\pi}{4}$ is unique in that it is numerically equal to the ordinate.

Second Method. We arbitrarily select a segment of line which extends from A to B, where B is such that its distance from the y-axis as measured by a rule placed on the graph is 20.0 millimeters. The height of B above the x-axis

is measured to be 22.2 millimeters. Then,

$$\tan\theta = \frac{22.2}{20.0} = 1.11.$$

The scale factor is

$$f = \frac{10 \text{ volts}}{\dfrac{\pi}{2}\text{ radian}}.$$

Hence, by Eq. (11–6)

$$m = 1.11\ \frac{10 \text{ volts}}{\dfrac{\pi}{2}\text{ radian}}$$
$$= 7.1 \text{ volts per radian.}$$

Exercise 11–5

1. What is the value of the slope of the curve $y = \frac{1}{2}x^2$ at $x = 2$; at $x = 0$?
2. Plot the curve $x^2 + y^2 = 25$. What type of curve is this? Determine the slope of this curve at points corresponding to the following values of x: -5, -3, 3, 5. (The slope at $x = -5$ and at $x = 5$ may be determined by inspection. For the remaining points (four altogether, two for each value of x) it is necessary to determine the slope graphically at only one point. The slope at the other points may be determined from the one computed slope together with a consideration of the symmetrical properties of the curve.)
3. Determine the slope of the curve $y = \sin x$ for the following values of x: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$, and π radians. Tabulate the results along with the corresponding value of $\cos x$ as indicated below:

x in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
slope of curve $y = \sin x$									
$\cos x$									

(Make graphical computations of slopes in only the first four cases. Obtain slopes in the last four cases by inferences from the results of the first four. Obtain the slope at $x = \frac{\pi}{2}$ by inspection.)

From the tabulated data plot a curve of cos x as a function of x and a curve of the slope of $y = \sin x$ as a function of x. Use the same set of coordinate axes for both curves.

4. Fill in the blanks in the following table:

x in degrees	0	30	45	60	90	120	135	150	180
slope of curve $y = \sin x$									
$\frac{\pi}{180} \cos x$									

From the tabulated data plot a curve of $y = \frac{\pi}{180} \cos x$ as a function of x and a curve of the slope of $y = \sin x$ as a function of x. Use the same set of coordinate axes for both curves.

5. Fill in the blanks in the following table:

θ in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
slope of curve y cos θ									
$-\sin \theta$									

Plot curves of these data using the same set of coordinates axes for both curves.

6. Determine for the following values of t: 0, 0.001, 0.002, 0.003, . . . 0.010 the slope of the curve $y = \sin \omega t$, where t denotes time in seconds and $\omega = 100\pi$ radians per second. Tabulate the results along with the corresponding values of $\omega \cos \omega t$ as indicated below:

t in thousandths of seconds	0	1	2	3	4	5	6	7	8	9	10
slope of curve $y = \sin \omega t$											
$\omega \cos \omega t$											

Plot curves of these data using the same set of coordinate axes for both curves.

7. The direct current (d-c) resistance of a circuit element is the ratio of the emf across the element to the current through the element. The variational, or alternating current (a-c), resistance of a circuit element is the reciprocal of the slope of the characteristic curve of current vs. voltage for the element. For a non-ohmic device both the d-c resistance and the a-c resistance are functions of the current or of the potential difference. Plot curves indicating

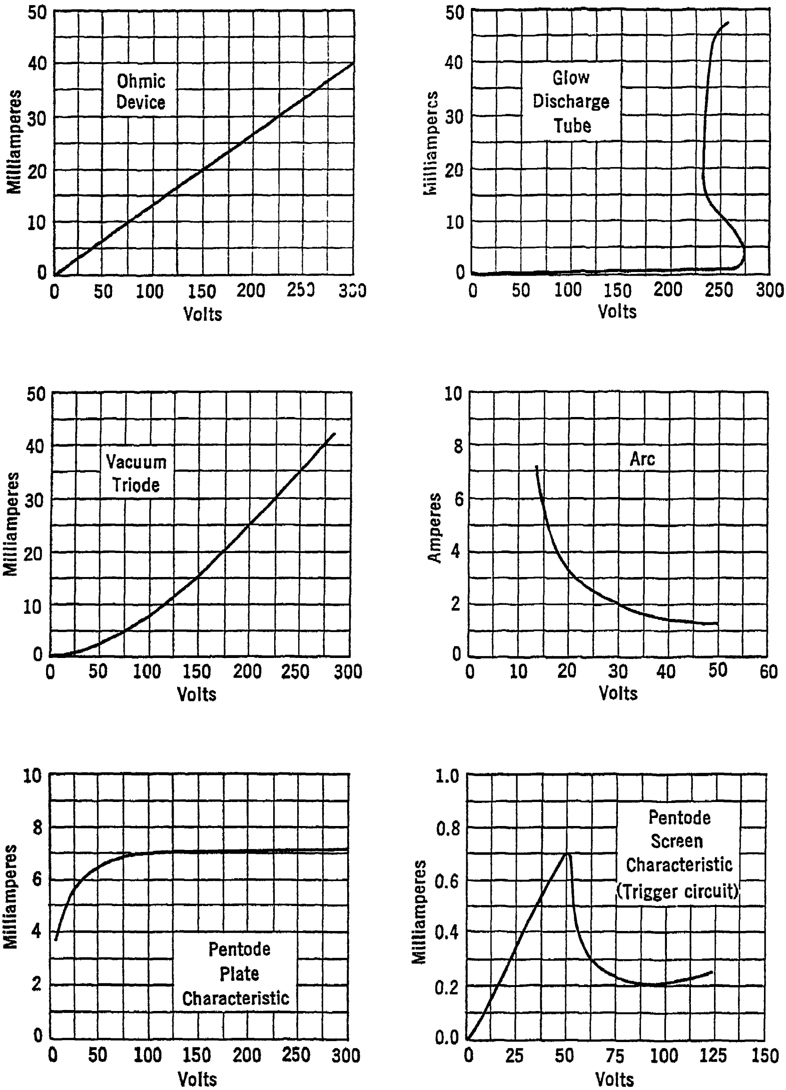


FIG. 11-11. Current-voltage characteristics of various devices.

the general trends of both d-c resistance and a-c resistance as functions of applied potential for each of the devices whose characteristics are shown in Fig. 11-11. Rough estimates of the slopes will suffice for the determination of a-c resistance in each case.

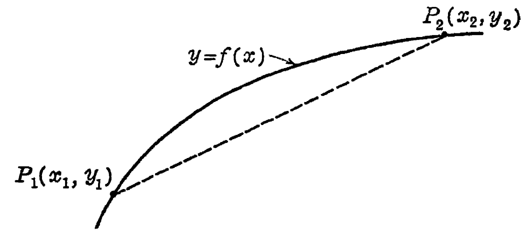


FIG. 11-12. Average rate of change of $f(x)$ with respect to x over the interval P_1P_2 is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.

11-7. Average Rate of Change. We define the *average rate of change* of a function, $f(x)$, with respect to x over an interval $P_1(x_1,y_1)$ to $P_2(x_2,y_2)$, as $\frac{y_2 - y_1}{x_2 - x_1}$ (see Fig. 11-12). The average rate of change of $f(x)$ with respect to x over the interval P_1P_2 is thus equal to the slope of the line P_1P_2.

11-8. Instantaneous Rate of Change. Let us denote any arbitrary abscissa interval $x_2 - x_1$ as Δx, and the corresponding ordinate interval $y_2 - y_1$ as Δy. We refer to Δx and Δy as *increments* of x and y, respectively. We now define the *instantaneous rate of change* of $f(x)$ with respect to x at a particular point $P_1(x_1,y_1)$ as the limit which is approached by $\frac{\Delta y}{\Delta x}$ as Δx approaches zero. In other words, *we define the instantaneous rate of change of $f(x)$ with respect to x at the point P_1 as the slope of the tangent line to the curve at the point P_1*, since, as Δx is taken successively smaller approaching zero (Fig. 11-13), the position of the chord P_1P_2 approaches coincidence with the tangent line and, hence, the *slope* of the chord approaches the *slope* of the tangent line

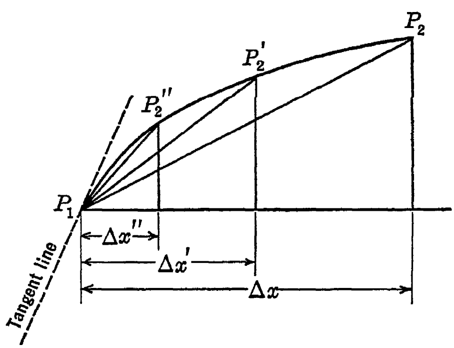


FIG. 11-13. The chord P_1P_2 approaches the tangent line.

Example 1. Determine the instantaneous rate of change of the function $y = \frac{1}{2}x^2$ with respect to x at the point $(-2,2)$.

In Example 1 of Sec. 11–6 we found that the slope of the tangent line to the curve $y = \frac{1}{2}x^2$ at the point $(-2,2)$ is -2. Hence, the instantaneous rate of change of the function $y = \frac{1}{2}x^2$ with respect to x at the point $(-2,2)$ is -2.

Example 2. Determine the instantaneous rate of change of the function $e = 10 \sin x$ volts (where x is measured in radians) at the point $\left(\frac{\pi}{4}, 7.1\right)$. In Example 2 of Sec. 11–6 we found that the slope of the curve $e = 10 \sin x$ volts at the point $\left(\frac{\pi}{4}, 7.1\right)$ is 7.1 volts per radian. Hence, the instantaneous rate of change of the function $e = 10 \sin x$ volts at the point $\left(\frac{\pi}{4}, 7.1\right)$ is 7.1 volts per radian.

Exercise 11–6

1. Find the instantaneous rate of change of $\frac{x^2}{2}$ with respect to x: (a) for $x = 2$, and (b) for $x = 0$.
2. Find the instantaneous rate of change of $\cos \theta$ with respect to θ (θ measured in radians) for the following values of θ: 0, $\frac{\pi}{6}$, and $\frac{\pi}{3}$ radians.
3. Find the instantaneous rate of change of the function $y = \frac{x^2}{2}$ with respect to x^2 for $x = 3$. (Regard x^2 as the independent variable, and write $x^2 = u$. The function is now $y = \frac{u}{2}$, and the problem is to find the instantaneous rate of change of $\frac{u}{2}$ with respect to u for $u = 9$. Plot $y = \frac{u}{2}$, and obtain the slope of this function at the point corresponding to $u = 9$.)

11–9. Derived Curve. Derivative. In general, the tangent line to a curve varies in direction along the curve, so that the slope of the tangent line is a function of the abscissa of the curve. If a function $\phi(x)$ is such that it is equal for each value of x to the slope of the graph of another function $f(x)$, then $\phi(x)$ is said to be *derived* from $f(x)$, and $\phi(x)$ is called the *derivative* of $f(x)$.

In Exercise 11–5 we plotted several derived functions, and (with allowances for errors in our graphical slope determinations) we noted similarities between the plotted *derived curves* and the curves of certain

functions for which the equations are known. For example, we observed a similarity (a) between the derived curve of $\sin x$ (with x in radians) and the curve of $y = \cos x$; (b) between the derived curve of $\sin x$ (with x in degrees) and the curve of $y = \frac{\pi}{180} \cos x$; (c) between the derived curve of $\cos \theta$ (with θ in radians) and the curve of $y = -\sin \theta$; and (d) between the derived curve of $\sin \omega t$ (with ωt in radians) and the curve of $\omega \cos \omega t$. It is possible to demonstrate (Chapter 22) that in each of the aforementioned cases the derived curve is represented exactly by the equation of the similar curve, or in other words: that the derivative of $\sin x$ (with x in radians) is $\cos x$; that the derivative of $\cos x$ (with x in radians) is $-\sin x$; that the derivative of $\sin x$ (with x in degrees) is $\frac{\pi}{180} \cos x$; and that the derivative of $\sin \omega t$ (with ωt in radians) is $\omega \cos \omega t$. For the present we shall not attempt to prove these relations but, instead, we shall confine our efforts to familiarizing ourselves with the graphical processes of obtaining derived curves.

Exercise 11-7

Plot each of the following functions and obtain derived curves therefrom. In each case use a scale of ordinates for the derived curve which is separate from that for the original curve.

1. $y = 2x$.
2. $y = 2x + 5$.
3. $y = 2x - 5$.
4. $y = \frac{1}{2}x^2$.
5. $y = \frac{1}{2}x^2 - 2$. What can you say about derivatives and derived curves for functions which differ only by a constant?
6. $y = \sin^2 x$.
7. $y = 10 \sin \theta + 2 \sin 3\theta$.

11-10. Notation. For a function $y = f(x)$ one may denote the derivative of y with respect to x by $\frac{dy}{dx}\cdot$ If $y = 3x^2$, the derivative is expressed as $\frac{d(3x^2)}{dx}\cdot$

The notation is suitably altered if letters other than x and y are employed as variables. Thus, for $u = z^2 + 2z$ the derivative of u with respect to z is expressed as $\frac{du}{dz}$ or $\frac{d(z^2 + 2z)}{dz}\cdot$ If $y = \sin \theta$, the deriva-

tive of y with respect to θ is denoted by $\frac{dy}{d\theta}$ or $\frac{d(\sin\theta)}{d\theta}$; and if $\theta = \omega t$, the derivative of y with respect to t is denoted by $\frac{dy}{dt}$ or $\frac{d(\sin\omega t)}{dt}$, wherein t rather than θ is regarded as the independent variable.

In place of a notation of the form $\frac{dy}{dx}$ or $\frac{d[f(x)]}{dx}$ any one of the following synonymous notations may also be employed: D_xy, $D_xf(x)$, y', $f'(x)$. The choice of notation in a particular case is usually based on convenience.

We shall ultimately (Sec. 25-1) define individual entities dy and dx, but for the present $\frac{dy}{dx}$ is to be regarded as a single quantity and not as the ratio of two quantities, dy and dx. The notation $\frac{d}{dx}y$ is often employed in place of $\frac{dy}{dx}$ in order to avoid the suggestion of a ratio which is characteristic of the notation $\frac{dy}{dx}\cdot$

11-11. Choice of the Radian as a Unit of Angle. Using the degree as the unit of angle, we find that

$$\frac{d(\sin\theta)}{d\theta} = \frac{\pi}{180}\cos\theta, \tag{11-7}$$

and

$$\frac{d(\cos\theta)}{d\theta} = -\frac{\pi}{180}\sin\theta. \tag{11-8}$$

On the other hand, using the radian as the unit of angle, we find that

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta, \tag{11-9}$$

and

$$\frac{d(\cos\theta)}{d\theta} = -\sin\theta. \tag{11-10}$$

The convenience of Eqs. (11-9) and (11-10) as compared with Eqs. (11-7) and (11-8), particularly in view of the fundamental nature and frequent

recurrence of $\frac{d(\sin\theta)}{d\theta}$ and $\frac{d(\cos\theta)}{d\theta}$, makes it advantageous to employ the radian as the unit of angle whenever working with derivatives and related concepts.

11-12. Requirements for the Existence of a Derivative. It is evident from the construction of derived curves that, in order for a function to have a *uniquely defined derivative* at every point in an interval, the curve of the function must be generally smooth throughout the interval. The

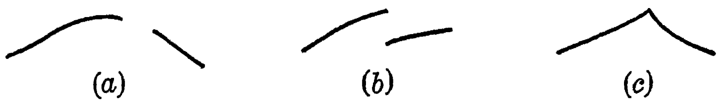


FIG. 11-14. Irregularities in a curve at which point the derivative is not uniquely defined.

curve cannot have a discontinuity, as in (a) or (b) of Fig. 11-14; it cannot have a sharp corner, or *cusp*, as in (c) of Fig. 11-14; it cannot be *multiple-valued*, as is the curve of $x^2 + y^2 = 25$ for $0 < |x| < 5$ (Problem 2 of Exercise 11-5); and it cannot have an infinite slope, as does the curve of $x^2 + y^2 = 25$ at $x = \pm 5$. This does not mean that we cannot draw derived curves for any but smooth given curves. However, with irregular curves we are obliged to omit portions of the derived curve which would correspond to irregularities in the original curve.

Exercise 11-8

Construct the derived curve corresponding to each of the following:

1. Square wave.
2. Isosceles triangular wave.

11-13. Second Derived Curve. Second Derivative. The derived curve of a derived curve is called a *second derived curve.* The function corresponding to a second derived curve is called a *second derivative.*

Exercise 11-9

Construct the second derived curve for the curves of each of the functions given in Problems 1, 4, and 6 of Exercise 11-7.

11-14. Integral Curve. Integral. If, of two curves a and b, b is the derived curve of a, then a is said to be the *integral curve* of b. The function corresponding to an integral curve is called an *integral.*

The process of constructing an integral curve from a given curve is the inverse process of constructing a derived curve from a given curve. The method is illustrated in Fig. 11-15, wherein the curve y is the integral curve for the curve y'. At each abscissa the slope of y is equal to the ordinate of y'. Thus, at $x = -1$ where $y' = 3$, the slope of y is 3; at $x = 0$ where $y' = 1$, the slope of y is 1; at $x = 1$ where $y' = -\frac{1}{2}$, the slope of y is $-\frac{1}{2}$; and at $x = 2$ where $y' = -1$, the slope of y is -1.

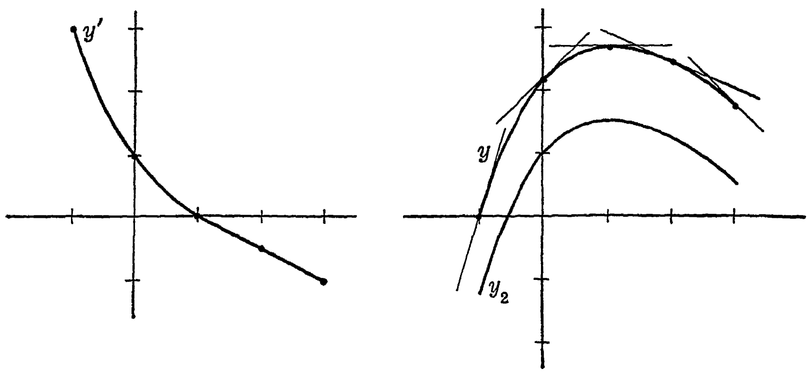


FIG. 11-15. Construction of integral curve (y) from given curve (y').

The given curve y' determines only the slope of the integral curve; hence, the curve labelled y_2 is an integral curve of y' since its slope is at each abscissa the same as that of y. This is consistent with our observations in Exercise 11-7 where we found the same derived curve for each of the curves associated with the functions of Problems 1, 2, and 3; and the same derived curve for each of the curves associated with the functions of Problems 4 and 5. We conclude that in drawing an integral curve the ordinate of the first point is optional.

Exercise 11-10

Construct integral curves corresponding to each of the following functions:

1. $y = 2$.
2. $y = x$.
3. $y = x + 1$.
4. $y = -x$.
5. $y = 2 \cos x$.
6. A square wave.
7. An isosceles triangular wave.

11-15. Calculus. That branch of mathematics which treats derivatives and their properties is known as *differential calculus* or, frequently (for

principally no better reason than tradition), as *the* differential calculus.

The following equations, which can be developed by the methods of calculus (Chapter 22), will be found immediately useful and are here presented without proof:

$$\frac{d(A \sin \theta + B)}{d\theta} = A \cos \theta; \tag{11–11}$$

$$\frac{d(A \cos \theta + B)}{d\theta} = -A \sin \theta; \tag{11–12}$$

$$\frac{d(A \sin \omega t + B)}{dt} = A\omega \cos \omega t; \tag{11–13}$$

$$\frac{d(A \cos \omega t + B)}{dt} = -A\omega \sin \omega t; \tag{11–14}$$

wherein ω, A, and B are any constants; and θ and t are any variables such that θ and ωt are measured in radians.

Exercise 11–11

Using Eqs. (11–11) through (11–14) find:

1. $\dfrac{d(\sin \alpha)}{d\alpha}$.
2. $\dfrac{d(\sin 2t)}{dt}$.
3. $\dfrac{d(\sin 3x)}{dx}$.
4. $\dfrac{d(2 \cos \beta)}{d\beta}$.
5. $\dfrac{d(I_m \sin \omega t)}{dt}$.
6. $\dfrac{d(50 \sin 377t)}{dt}$.
7. $\dfrac{d(E_m \cos \omega t)}{dt}$.

CHAPTER 12

SOLUTION OF TRIANGLES

12–1. Nomenclature. The three angles and the three sides of a triangle are called its *parts*. If the magnitudes of some of the parts of a triangle are known, it is frequently possible to determine the remaining parts. This process is known as *solving the triangle*. In this chapter each angle of a triangle is designated by a capital letter, and the side opposite is designated by the corresponding small letter.

12–2. Right Triangles. The solution of right triangles is illustrated by the following examples:

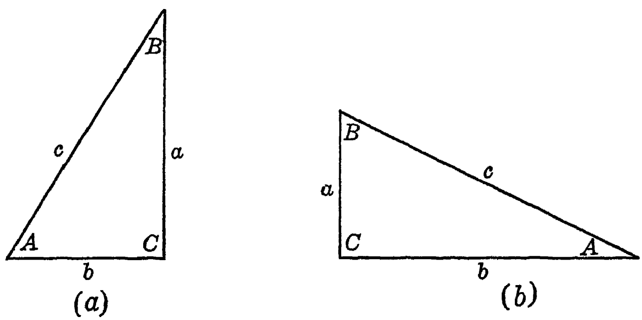


FIG. 12–1. Right Triangles.

Example 1. Solve the right triangle ABC of Fig. 12–1(a) if $a = 5$ and $B = 25°$. In Fig. 12–1(a),

$$A = 90° - B = 90° - 25° = 65°.$$

$$\frac{b}{a} = \tan B,$$

or

$$b = a \tan B = 5 \tan 25°.$$

From Table 9–1,

$$\tan 25° = 0.466;$$

hence,

$$b = 5 \cdot 0.466 = 2.33.$$

$$\frac{a}{c} = \cos B,$$

or

$$c = \frac{a}{\cos B} = \frac{5}{\cos 25°}.$$

From Table 9–1,

$$\cos 25° = 0.906;$$

hence,

$$c = \frac{5}{0.906} = 5.52.$$

Example 2. Solve the right triangle ABC of Fig. 12–1(b) if $a = 6$ and $b = 10$. In Fig. 12–1(b),

$$\tan A = \frac{a}{b} = \frac{6}{10}.$$

From Table 9–1,

$$\tan^{-1} 0.6 = 31°;$$

hence,

$$A = 31°.$$

$$B = 90° - A = 90° - 31° = 59°.$$

$$\frac{a}{c} = \sin A,$$

or

$$c = \frac{a}{\sin A}.$$

From Table 9–1,

$$\sin 31° = 0.515;$$

hence,

$$c = \frac{6}{0.515} = 11.7.$$

Exercise 12-1

Solve the right triangle ABC, in which $C = 90°$, given:

1. $a = 3, b = 4$.
2. $b = 4, B = 10°$.
3. $a = 6, B = 42°$.
4. $c = 25, A = 35°$.
5. $b = 12, c = 13$.
6. $c = 3, A = 55°$.
7. $a = 20, B = 47°$.
8. $a = 36, B = 50°$.
9. $b = 40, A = 31°$.
10. $b = 10, A = 72°$.

12-3. Oblique Triangles. The triangle of Fig. 12-2(a) has one obtuse angle; the triangle of Fig. 12-2(b) has all acute angles. For both triangles, or in other words, for oblique triangles in general, it may be shown that:

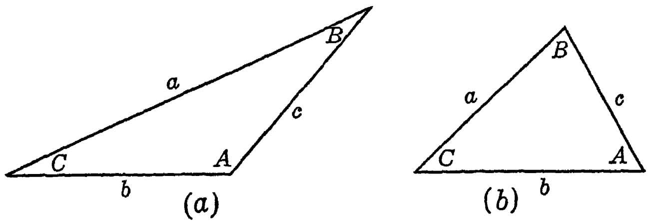


FIG. 12-2. (a) Triangle with one obtuse angle; (b) triangle with all acute angles.

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A, \\ b^2 &= c^2 + a^2 - 2ca \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C; \end{aligned}\right\} \tag{12-1}$$

and that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{12-2}$$

The relations in Eq. (12-1) are known as the *Law of Cosines*, and the relations in Eq. (12-2) are known as the *Law of Sines*.

A method of proof for the Law of Cosines, Eq. (12-1), is as follows. With reference to Fig. 12-3(a),

$$\begin{aligned} a^2 &= (b + d)^2 + e^2 = (b^2 + 2bd + d^2) + e^2 \\ &= (b^2 + 2bd + d^2) + c^2 - d^2 = b^2 + 2bd + c^2. \end{aligned}$$

But

$$\frac{d}{c} = \cos \theta,$$

so that

$$d = c \cos \theta.$$

Hence,

$$a^2 = b^2 + c^2 + 2bc \cos \theta;$$

and since

$$\cos \theta = \cos (180° - A) = -\cos A,$$
$$a^2 = b^2 + c^2 - 2bc \cos A.$$

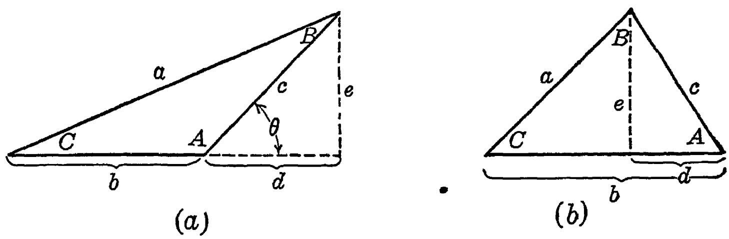


FIG. 12-3. Construction for proving the Law of Cosines and the Law of Sines.

With reference to Fig. 12-3(b),

$$\begin{aligned} a^2 &= (b - d)^2 + e^2 = (b^2 - 2bd + d^2) + e^2 \\ &= (b^2 - 2bd + d^2) + c^2 - d^2 = b^2 - 2bd + c^2. \end{aligned}$$

But

$$\frac{d}{c} = \cos A,$$

so that

$$d = c \cos A.$$

Hence,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

which is the same result as that obtained for the triangle of Fig. 12-3(a).

By similar arguments, on drawing perpendiculars from the vertices A and C to the opposite sides, respectively, we can show for both types of triangle that

$$b^2 = c^2 + a^2 - 2ca \cos B,$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

A method of proof for the Law of Sines, Eq. (12–2), is as follows. With reference to Fig. 12–3(a),

$$\frac{e}{a} = \sin C,$$

or

$$e = a \sin C. \tag{12–3}$$

Also,

$$\frac{e}{c} = \sin \theta = \sin (180° - A) = \sin A,$$

or

$$e = c \sin A. \tag{12–4}$$

From Eqs. (12–3) and (12–4),

$$a \sin C = c \sin A.$$

With reference to Fig. 12–3(b),

$$\frac{e}{a} = \sin C,$$

or

$$e = a \sin C. \tag{12–5}$$

Also,

$$\frac{e}{c} = \sin A,$$

or

$$e = c \sin A. \tag{12–6}$$

From Eqs. (12–5) and (12–6),

$$a \sin C = c \sin A.$$

In both cases we have the same result, $a \sin C = c \sin A$, or

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \tag{12–7}$$

By similar arguments, on drawing perpendiculars from the vertices A and C to the opposite sides, respectively, we can show for both types

of triangle that

$$\frac{b}{\sin B} = \frac{c}{\sin C} \tag{12–8}$$

and

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{12–9}$$

Eq. (12–2) follows from Eqs. (12–7), (12–8) and (12–9).

Exercise 12–2

A. Give in detail the argument to prove that for both triangles in Fig. 12–3:

1. $b^2 = c^2 + a^2 - 2ca \cos B$ (Law of Cosines). Suggestion:

For the case of the triangle of Fig. 12–3(a) erect a perpendicular from A to the opposite side as shown in Fig. 12–4. With reference to Fig. 12–4,

$$\begin{aligned} b^2 &= d^2 + (a - e)^2 = d^2 + (a^2 - 2ae + e^2) \\ &= c^2 - e^2 + (a^2 - 2ae + e^2) = c^2 + a^2 - 2ae. \end{aligned}$$

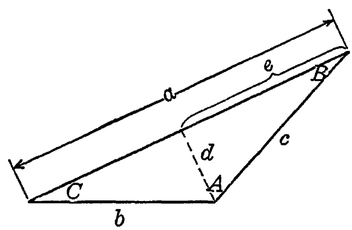


FIG. 12–4. Construction for proving the Law of Cosines and the Law of Sines.

But,

$$\cos B = \frac{e}{c}.$$

or

$$e = c \cos B.$$

For the case of the triangle of Fig. 12–3(b) erect a perpendicular from A to the opposite side.

2. $c^2 = a^2 + b^2 - 2ab \cos C$ (Law of Cosines).

3. $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (Law of Sines).

4. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ (Law of Sines).

B. Values of parts of a triangle are given in each of the following. Solve for the values of the remaining parts:

1. $a = 12$, $A = 25°$, $C = 37°$.

2. $a = 4$, $b = 3$, $A = 75°$.

3. $a = 7$, $c = 8$, $B = 60°$.

4. $a = 2$, $b = 1$, $C = 10°$.

5. $a = 7$, $b = 5$, $c = 3$.

CHAPTER 13

VECTORS

13–1. Definition of Vector. Certain quantities can be completely described only in terms of a magnitude and a direction. Forces are among such quantities. Common units of measure of forces are the pound weight, the kilogram weight, and the dyne. The complete measure of a force which acts at some particular point is described by specifying its magnitude, for example, 5 pounds, together with its direction, for example, horizontal and 20° north of east. Pictorially, such a force may be represented by a directed line segment, or vector, as in Fig. 13–1. Here the length of the line segment is taken as 5 units (on an arbitrary scale of 1 unit of length representing 1 pound force), and the direction of the line is shown in the conventional mapping manner for 20° north of east. The fact that the force is horizontal cannot be shown in a simple two-dimensional portrayal, such as that of Fig. 13–1, but for many engineering applications two-dimensional portrayal of directed quantities is adequate.

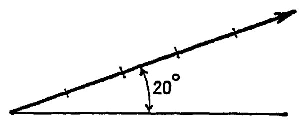


FIG. 13–1. Vectorial representation of a force.

A directed quantity, such as the force illustrated above, is called a *vector*. It must not be construed from the use of force as an illustration of a vector that all vectors are forces. A vector is simply any quantity which is described by a magnitude and a direction.

A quantity which is independent of direction, for example, the resistance of a wire, or the horsepower of a motor, or simply an ordinary number, is referred to as a *scalar* to distinguish it from a vector. Most of the quantities with which we have heretofore been concerned in this text would be classified as scalars.

Exercise 13–1

Designate each of the following as scalar or vector:

1. Temperature. **2.** Volume. **3.** Wind velocity.

13-2. Vector Representation. In this text we shall employ a bold face letter to denote a vector. In a few cases where it is desirable to describe a vector by specifying its end points, we shall employ a pair of bold face letters; thus: **OA**, where O represents the tail of the vector, and A, the head. Sometimes, when the implication is clear, no special symbolism will be employed to designate vectors.

Although it is commonly used in the literature, the bold face notation is not adaptable to written work by the individual. It is suggested that either a dot over a letter or an arrow over a letter be used by the student to designate a vector in his own writing.

13-3. Magnitude and Direction. A vector pictorially represents magnitude and direction only. It is not intended that a vector should represent, in addition, line of action or point of application of a physical quantity. This means that a vector may be displaced parallel to itself

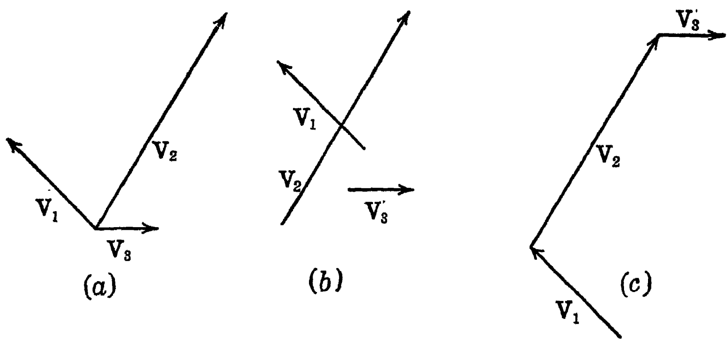


FIG. 13-2. Equivalent sets of vectors.

without altering its interpretation, since in its displaced position, its magnitude and direction are unchanged. Vectors are sometimes shown starting from a common origin, as in Fig. 13-2(a). Equivalent representations are those shown in Fig. 13-2(b) and (c), wherein each of the vectors of (a) has been displaced.

The magnitude of a vector is called the *absolute value* of the vector. The term " absolute value " meaning magnitude of the vector without regard to direction is a natural extension of the term " absolute value " as it is employed with scalars to designate magnitude without regard to sign. The absolute value of a vector **A** is designated $|\mathbf{A}|$ just as the absolute value of a scalar A is designated $|A|$.

The direction of a vector may be specified relative to any definite reference line. However, unless a particular reference line is designated, it is conventional in electrical practice to describe the direction of a vector in terms of that angle through which the positive x-axis must be rotated in a counterclockwise direction about the origin in order to become parallel to (or coincident with), and in the same sense of direction as, the given vector. For any given vector we shall refer to this angle as the *angle of the vector*. In Fig. 13–3 the angles of the vectors **A**, **B**, and **C** are 30°, 135°, and 300°, respectively. In terms of clockwise (negative) rotation the angles are $-330°$, $-225°$, and $-60°$.

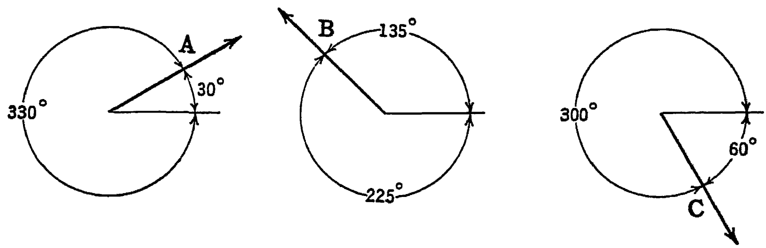


FIG. 13–3. Vectors of various angles.

One method of describing a vector in a plane (two dimensional space) is to state magnitude, M, and angle, θ, as in the following: $M\underline{/\theta}$, $5\underline{/40°}$, $12\underline{/7°}$. Sometimes the symbol $\overline{\diagdown}$ is used to denote a negative angle. With this notation $3\underline{/-20°}$ and $3\overline{\backslash 20°}$ are identical vectors. Vectors $M\underline{/\theta}$ and $M\underline{/\theta \pm 360°}$ are likewise identical.

A directed physical quantity in three-dimensional space requires at least two angles (such as one meridian angle and one azimuth angle). as well as a magnitude to specify it completely. We shall not here, however, consider other than plane vectors.

Exercise 13–2

Draw the following vectors:

1. $4\underline{/40°}$.
2. $12\underline{/-7°}$.
3. $0.5\underline{/270°}$.
4. $2\overline{\backslash 60°}$.
5. $2.5\underline{/420°}$.

13–4. Vector Addition. It is observed to be the case with a certain class of physical phenomena that, where each of two independent effects

may be represented by individual vectors, $\mathbf{V}_1$ and $\mathbf{V}_2$, the combined effect is properly represented in magnitude and in direction by a third vector, $\mathbf{V}_3$, which is formed by the diagonal of the parallelogram constructed from the first two vectors as sides (Fig. 13-4).

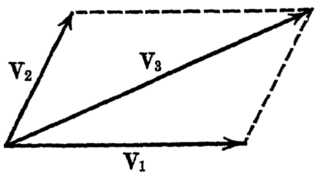


FIG. 13-4. $\mathbf{V}_3$ equivalent to both $\mathbf{V}_1$ and $\mathbf{V}_2$.

Among quantities which behave in the foregoing manner are displacements, velocities, accelerations, and forces. Included in forces are electric and magnetic forces.

With reference to Fig. 13-4 we find it advantageous to speak of $\mathbf{V}_3$ as the *vecto srum* of $\mathbf{V}_1$ and $\mathbf{V}_2$ and to write

$$\mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2.$$

Graphically, then, the addition of two vectors consists of drawing the diagonal of the parallelogram which is formed by using the two given vectors for two sides.

Exercise 13-3

Add the following vectors graphically:

1. $5\underline{/0°}$, $15\underline{/30°}$.
2. $2\underline{/45°}$, $2\underline{/-45°}$.
3. $3\underline{/90°}$, $4\underline{/235°}$.
4. $5\underline{/0°}$, $2\underline{/0°}$. (Regard as the limiting case of a parallelogram.)
5. $2\underline{/0°}$, $5\underline{/180°}$.
6. $2\underline{/103°}$, $7\underline{/103°}$.

13-5. Alternative Scheme of Vector Addition. The same vector relations as in Fig. 13-4 might have been obtained by regarding $\mathbf{V}_3$ as the third side of a triangle formed from $\mathbf{V}_1$ and $\mathbf{V}_2$ as the first two sides. This scheme, as indicated in Fig. 13-5(a), is known as the *triangle method,* or the *head-to-tail method,* of vector addition. This technique is particularly suitable for graphical addition of several vectors as shown in Fig. 13-5(b).

Once a problem in vector addition is graphed, either by the parallelogram method or by the triangle method, the solution is then simply a matter of determining by any means whatsoever (a) the length and (b) the direction of that arrow which represents the desired vector

quantity. Usually it happens that this determination can be accomplished accurately by analytical methods (Sec. 13–7 and 13–8). Frequently in practical problems, however, it is sufficient, or even desirable (for economy of time), to obtain the length of a particular arrow by direct measurement with a rule (converting to appropriate units as

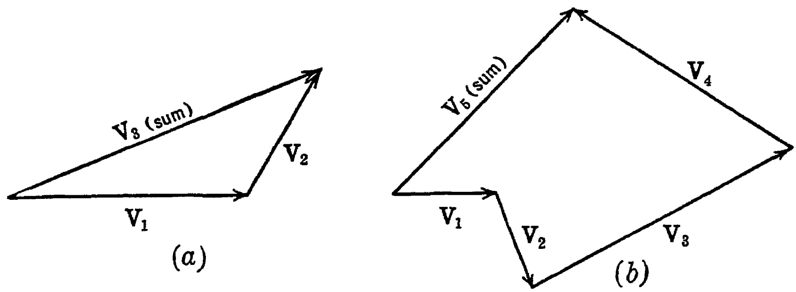


FIG. 13–5. Head-to-tail method of vector addition.

necessary) and to obtain the angle with a protractor. This is called a graphical solution. Even when the requirements of precision necessitate an analytical solution, it is advantageous to obtain at least a rough graphical solution to serve as a check upon the reasonableness of the numerical calculations.

Exercise 13–4

Add the following vectors graphically by the head-to-tail method:

1. $5\underline{/270^\circ}$, $12\underline{/180^\circ}$.
2. $10\underline{/15^\circ}$, $10\underline{/45^\circ}$.
3. $4\underline{/30^\circ}$, $3\underline{/60^\circ}$, $8\underline{/225^\circ}$.
4. $10\underline{/90^\circ}$, $5\underline{/210^\circ}$, $8\underline{/0^\circ}$.

13–6. Components. The sum of two or more given vectors is called the *resultant* of the given vectors. Two or more vectors which add to yield a given vector are called *components* of the given vector. In Fig. 13–5(a) $\mathbf{V}_3$ is the resultant of $\mathbf{V}_1$ and $\mathbf{V}_2$; and $\mathbf{V}_1$ and $\mathbf{V}_2$ are components of $\mathbf{V}_3$. In Fig. 13–5(b), $\mathbf{V}_5$ is the resultant of $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, and $\mathbf{V}_4$; and $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, and $\mathbf{V}_4$ are components of $\mathbf{V}_5$.

It is often useful to *resolve* (decompose) *a vector* into two mutually perpendicular components. In Fig. 13–6(a) is sketched a railway car which is being towed by a tractor from the roadway alongside the track. The force exerted by the cable on the car is represented in magnitude and direction by **C** in Fig. 13–6(b). That component of the cable tension

which is effective in pulling the car along the track is represented in magnitude and direction by **A**. And that component of the cable tension

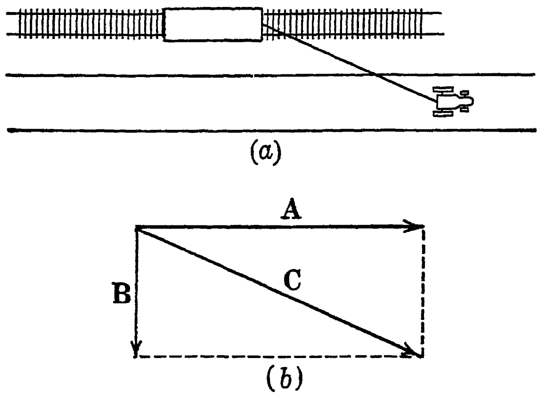


FIG. 13-6. (a) Tractor pulling railway car; (b) vector diagram of mutually perpendicular force components.

which tends to pull the car toward one side of the track is represented in magnitude and direction by **B**.

Exercise 13-5

A. Arrange graphically vectors of the following magnitudes so that their resultant is zero:

1. 6, 6. **2.** 5, 12, 13. **3.** 4, 4, 4.

B. Resolve graphically each of the following vectors into two mutually perpendicular components, one component to be at an angle of either 45° or 225°.

1. $3\underline{/60^\circ}$. **2.** $10\underline{/180^\circ}$. **3.** $5\underline{/-30^\circ}$.

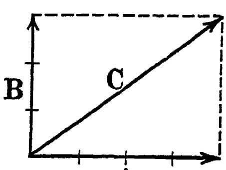


FIG. 13-7. Addition of two perpendicular vectors.

13-7. Addition of Perpendicular Vectors Analytically. Analytical computation in vector addition may be best illustrated with an example. Let us suppose that it is desired to add the vectors $4\underline{/0^\circ}$ and $3\underline{/90^\circ}$. These vectors are **A** and **B**, respectively, in Fig. 13-7. Completing the parallelogram, which in this special instance is a rectangle, we obtain **C** as the vector sum. The magnitude of **C** may be found by means of the

Pythagorean Theorem as $\sqrt{4^2 + 3^2}$ or 5. θ, the angle of **C**, may be found from the right triangle relation, $\sin\theta = \frac{3}{5}$. From Table 9-1, $\theta = 36^\circ\ 50'$.

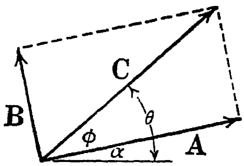


FIG. 13-8. Addition of two perpendicular vectors, general case.

The vectors **A** and **B** in Fig. 13-8 represent the general case of two vectors which are perpendicular to each other. Here the magnitude of **C** is obtained from the magnitudes of **A** and **B** just as in the preceding example. And the angle of **C** is found from $\alpha + \phi$, where α is the angle of **A** and ϕ is the angle between **A** and **C**.

Exercise 13-6

A. Add analytically the following vectors:

1. $3\underline{/0^\circ}$, $6\underline{/90^\circ}$.
2. $5\underline{/270^\circ}$, $12\underline{/180^\circ}$.
3. $10\underline{/-45^\circ}$, $10\underline{/45^\circ}$.
4. $2\underline{/120^\circ}$, $3\underline{/210^\circ}$.

B. Resolve analytically each of the following vectors into two mutually perpendicular components, one component to be at an angle of either 45° or 225° (compare part *B* of Exercise 13-5):

1. $3\underline{/60^\circ}$.
2. $10\underline{/180^\circ}$.
3. $5\underline{/-30^\circ}$.

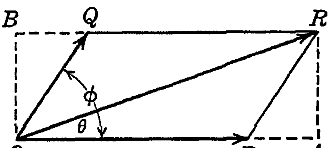


FIG. 13-9. Addition of two non-perpendicular vectors.

13-8. Addition of Two Non-Perpendicular Vectors Analytically. The addition of two non-perpendicular vectors may be accomplished by an extension of the method for two perpendicular vectors. Thus, let us suppose it is desired to add the two vectors shown in Fig. 13-9 or, what amounts to the same thing, to find the length and direction of **OR**. Let us construct a rectangle *OARB* (Fig. 13-9) of which **OR** is a diagonal. With the aid of this rectangle we can then reduce the problem to one of adding perpendicular vectors. We see from the diagram of Fig. 13-9 that angle RPA = angle QOP, that is, ϕ. Thus:

$$\frac{\overline{AR}}{\overline{PR}} = \sin\phi,$$

from which

$$\overline{AR} = \overline{PR}\sin\phi;$$

and

$$\frac{\overline{PA}}{\overline{PR}} = \cos\phi,$$

from which

$$\overline{PA} = \overline{PR}\cos\phi.$$

We can obtain the magnitude of **OR** from

$$\overline{OR}^2 = \overline{OA}^2 + \overline{AR}^2;$$

where

$$\overline{OA} = \overline{OP} + \overline{PA};$$

and we can obtain the direction of **OR** from

$$\frac{\overline{AR}}{\overline{OR}} = \sin\theta.$$

In essence what we did in this example was (1) to resolve **OQ** into two mutually perpendicular components, **PA** and **AR**; and then (2) to add vectorially **OP**, **PA**, and **AR**. This point of view is represented in the diagram of Fig. 13–10. In (a) **OQ** is resolved into two components, one horizontal and one vertical. In (b) the horizontal component of **OQ** is added to **OP**. In (c) the vertical component of **OQ** is added to the sum of the horizontal vectors of (b) to yield the resultant **OR**.

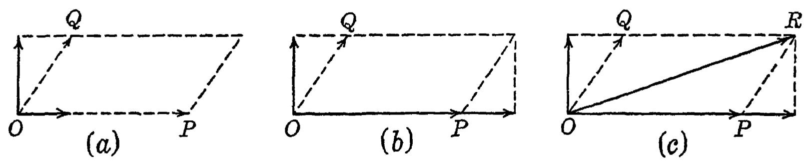


FIG. 13–10. Steps in the addition of two non-perpendicular vectors.

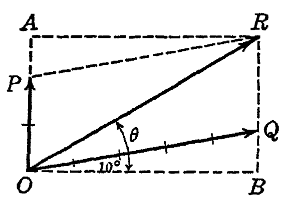


FIG. 13–11. Addition of two non-perpendicular vectors.

13–9. Illustration of the Analytical Method with a Numerical Case. For a numerical case which is to be solved analytically let us consider the following:

Example. Find the sum of $2\underline{/90^\circ}$ and $5\underline{/10^\circ}$.

First Method. The given vectors and the auxil-

iary diagram are drawn in Fig. 13–11. Here $\overline{BQ}$, the magnitude of the vertical component of **OQ**, is

$$5 \sin 10° = 5 \cdot 0.174 = 0.870;$$

and $\overline{OB}$, the magnitude of the horizontal component of **OQ**, is

$$5 \cos 10° = 5 \cdot 0.985 = 4.93.$$

Then the magnitude of the desired sum vector **OR** is

$$|\mathbf{OR}| = \overline{OR} = \sqrt{\overline{OB}^2 + \overline{BR}^2} = \sqrt{4.93^2 + (0.870 + 2)^2} = 5.71;$$

and θ, the angle of **OR** is given by

$$\cos \theta = \frac{\overline{OB}}{\overline{OR}} = \frac{4.93}{5.71} = 0.862.$$

$$\theta = 30°\, 30'.$$

Second Method. The problem may be solved by using the Law of Sines and the Law of Cosines. Referring to Fig. 13–11, in the triangle OQR we have by Eq. (12–1):

$$\overline{OR}^2 = \overline{OQ}^2 + \overline{QR}^2 - 2\overline{OQ} \cdot \overline{QR} \cdot \cos (\angle\, OQR).$$

$\angle\, OQR$ is seen to be equal to 100°, since

$$\angle\, OQB = 90° - 10° = 80°,$$

and

$$\angle\, OQR = 180° - \angle\, OQB.$$

Thus,

$$\begin{aligned} \overline{OR}^2 &= 5^2 + 2^2 - 2 \cdot 5 \cdot 2 \cdot \cos 100° \\ &= 25 + 4 + 20 \cdot \sin 10° \\ &= 32.48. \end{aligned}$$

The magnitude of **OR** is then

$$|\mathbf{OR}| = \overline{OR} = \sqrt{32.48} = 5.70.$$

By Eq. (12–2),

$$\frac{\overline{QR}}{\sin (\angle\, ROQ)} = \frac{\overline{OR}}{\sin (\angle\, OQR)},$$

or

$$\frac{2}{\sin (\angle\, ROQ)} = \frac{5.70}{\sin 100°},$$

from which

$$\sin(\angle ROQ) = \frac{2}{5.70}\sin 100^\circ = \frac{2}{5.70}\sin 80^\circ$$

$$= \frac{2}{5.70}\cdot 0.985 = 0.346.$$

$$\angle ROQ = 20^\circ\, 10';$$

so that the angle of **OR** is

$$\theta = \angle ROQ + 10^\circ = 30^\circ\, 10'.$$

The slightly different results by the two methods (5.71 and 30° 30′ as compared with 5.70 and 30° 10′) is a consequence of our limiting sine and cosine values to three decimal places, and angles to 10′ subdivisions.

Exercise 13–7

Add analytically the following vectors:

1. $5\underline{/0^\circ}$, $15\underline{/30^\circ}$.
2. $10\underline{/15^\circ}$, $10\underline{/45^\circ}$.
3. $2\underline{/60^\circ}$, $4\underline{/210^\circ}$.
4. $4\underline{/30^\circ}$, $3\underline{/60^\circ}$, $8\underline{/225^\circ}$. (Resolve each vector into components along two mutually perpendicular axes, one axis coinciding with any one of the three given vectors. Add the components along each of the axes individually.)
5. $10\underline{/90^\circ}$, $5\underline{/210^\circ}$, $8\underline{/0^\circ}$.

13–10. Vector Subtraction. The diagram of Fig. 13–11 might have been used had it been required to find the vector difference of **OR** and **OQ**. The problem in this case would be one of finding the magnitude and direction of **OP** from the given magnitude and direction of **OR** and **OQ**. In general, by the *vector difference* of two vectors, **A** − **B**, is meant that vector **C** which when added to **B** equals **A**. **A** − **B** = **C** if **B** + **C** = **A**. In other words, from one standpoint, the operation **A** − **B** amounts to finding that vector **C** which with **B** forms a parallelogram of which **A** is the diagonal. From another viewpoint **A** − **B** may be regarded as the addition to **A** of a vector −**B**, where −**B** is a vector which is in magnitude equal to **B**, but in the opposite direction. Relative to **B** we may speak of −**B** as a *negative vector*.

Exercise 13-8

Find the indicated vector differences:

1. $12\underline{/90^\circ} - 2\underline{/90^\circ}$.
2. $2\underline{/90^\circ} - 12\underline{/90^\circ}$.
3. $5\underline{/45^\circ} - 14\underline{/225^\circ}$.
4. $10\underline{/60^\circ} - 10\underline{/90^\circ}$.
5. $10\underline{/90^\circ} - 10\underline{/60^\circ}$.
6. $3\underline{/60^\circ} - 2\underline{/210^\circ}$.
7. $8\underline{/135^\circ} - 2\underline{/225^\circ}$.
8. $2\underline{/135^\circ} - 8\underline{/225^\circ}$.

13-11. Multiplication and Division of a Vector by a Scalar. In Fig. 13-12 are shown a vector, $\mathbf{V}_1$, and another vector, $\mathbf{V}_2$, the latter being three times the former in magnitude and in the same direction as the former. It is natural to define the *product of a vector by a positive scalar* as an extension of vector addition — the addition of a vector to itself — so that in this instance we write

FIG. 13-12. Multiplication of a vector by a scalar.

$$\mathbf{V}_2 = 3\mathbf{V}_1.$$

By the *product of a negative number and a vector* we shall mean the product of the corresponding positive number and negative vector. Thus $-3\mathbf{V} = 3 \cdot (-\mathbf{V})$, where $-\mathbf{V}$ is in magnitude equal to $\mathbf{V}$ but in the opposite direction. The *division of a vector by a scalar* involves finding that vector which when added to itself an appropriate number of times yields the original vector. The multiplication, or division, of a vector by a scalar produces a new vector in the same direction as the original vector but of altered magnitude. The magnitude of the resultant vector is obtained by multiplying, or dividing (as the case may be), the magnitude of the given vector by the magnitude of the scalar.

Exercise 13-9

Perform the indicated operations:

1. $5 \cdot (10\underline{/33^\circ})$.
2. $-5 \cdot (10\underline{/33^\circ})$.
3. $(8\underline{/-7^\circ}) \cdot 3$.

13-12. Complex Number Vector. Certain electrical quantities are conveniently treated through the use of vector methods although the electrical quantities themselves may not be vectors. For example, in alternating current studies it is useful to represent resistance by a vector directed horizontally, and to represent current by a vector which rotates. In the following material we shall employ the notion of a plane vector

not as a representation of a physical directed quantity but simply as an abstract convenience in graphical studies of alternating current circuits. This latter type of vector (or pseudo-vector) we sometimes refer to as a complex number vector because of its relation to the complex number system (Sec. 15–12).

Exercise 13–10

1. For the series circuit of Fig. 13–13(a) the resistance may be represented as in Fig. 13–13(b) by a horizontal vector directed to the right. Then the inductive reactance is represented by an upward pointing vertical vector, and

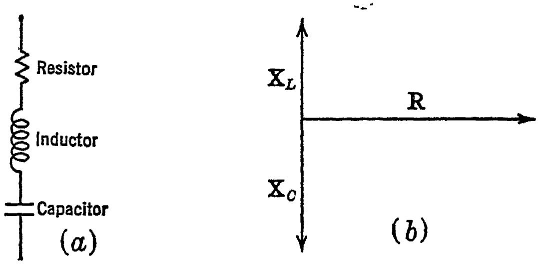


FIG. 13–13. (a) Resistance-inductance-capacitance series circuit; (b) Vector diagram for circuit shown in (a).

the capacitive reactance is represented by a downward pointing vertical vector. The impedance of this circuit is represented by the vector sum of these three vectors.

Find the magnitude and angle of the impedance vector for the circuit of Fig. 13–13(a) if the magnitude of the resistance is 4 ohms, the magnitude of the inductive reactance is 2 ohms, and the magnitude of the capacitive reactance is 3 ohms.

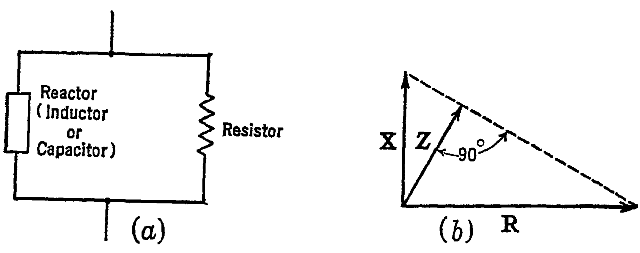


FIG. 13–14. (a) Resistance-reactance parallel circuit; (b) Vector diagram for circuit shown in (a).

2. Fig. 13–14(b) shows a graphical method for determining the vector which

represents the impedance, **Z**, of a circuit consisting of a resistance, **R**, and a reactance, **X**, in parallel.

Find the magnitude and angle of the impedance of the parallel circuit of Fig. 13–14(a) in which the reactance is capacitive (vector vertically down) of amount 3 ohms, and the resistance is in magnitude equal to 8 ohms.

CHAPTER 14

THE ROTATING VECTOR

14–1. Development of the Sine Curve. In Fig. 14–1 is shown a vector of unit length which rotates at a uniform rate in a counterclockwise (the conventionally positive) direction. Since the length of the vector is 1, the vertical projection, y, of the vector may be considered to represent the instantaneous magnitude of the sine of the angle θ $\left(\sin\theta = \dfrac{y}{1}\right)$.

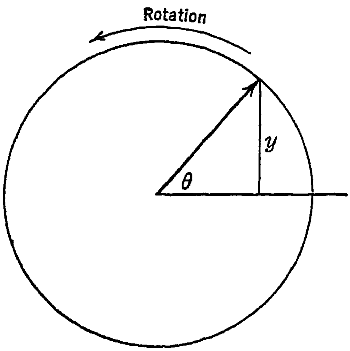


FIG. 14–1. Rotating unit vector; $y = \sin\theta$.

In Fig. 14–2(a) the projection is developed into a sine wave by horizontally displacing the position of the projection as the vector rotates. When the vector begins rotation, it is pointing horizontally to the right, and its projection is at

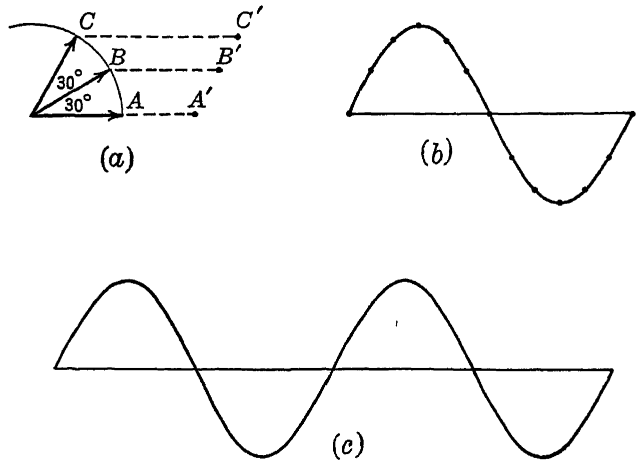


FIG. 14–2. Development of sine curve.

A'. After the vector has rotated through 30°, the arrowhead is at B, and the projection is at B'. Rotation of the vector through another 30° brings the arrowhead to C and the projection to C'. A complete rotation of the vector yields the corresponding curve traced by the projection as shown in Fig. 14-2(b); and continuous rotation produces the repeated sine wave of Fig. 14-2(c).

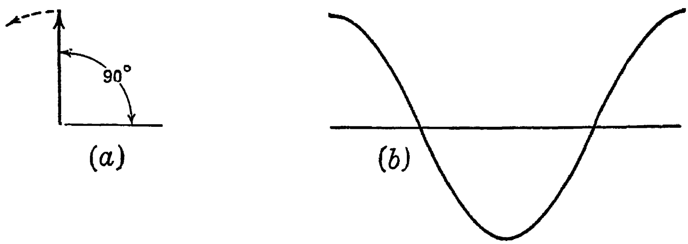


FIG. 14-3. Development of sine curve; generating vector initially vertically up.

Should the vector begin its rotation, as in Fig. 14-3(a), from an initial vertical-and-up position, the accompanying sine* curve is as in Fig. 14-3(b).

14-2. Phase. The magnitude of the angle of rotation at any particular instant is referred to as the *phase* of the motion at that instant. If the vectors of Figs. 14-2 and 14-3 rotate with the same uniform velocity, then the phase difference in their motions is constant and equal to 90°. Under these circumstances the vector and the associated sine curve of Fig. 14-3 are said to *lead* the vector and the sine curve of Fig. 14-2 by 90°; or, equally well, the latter are said to *lag* the former by 90°.

Exercise 14-1

$\mathbf{V}_1$ and $\mathbf{V}_2$ are rotating vectors each of unit magnitude. $\mathbf{V}_1$ leads $\mathbf{V}_2$ in phase by 30°. Draw a diagram showing the positions of both vectors when the phase of $\mathbf{V}_1$ is (a) 0°, (b) 30°, (c) 180°.

14-3. Amplitude. If in Fig. 14-2 the length of the rotating vector had been 2 instead of 1, the vertical projection of the vector for any angle θ

* This curve is actually a cosine curve. However, in the general sense, it may be regarded as a sine curve displaced through an angle of 90°. In a similar manner we also speak of any curves which are generated by the projection of rotating vectors as "sine" curves.

would have been $2 \sin \theta$ since we should then have had $\sin \theta = \frac{y}{2}$ or $y = 2 \sin \theta$. The development in the manner of Fig. 14-2, using a rotating vector of length 2, would thus have yielded a curve whose ordinate at each point is equal to $2 \sin \theta$. The two rotating vectors, of length 1 and 2, respectively, and their associated sine curves, $y = \sin \theta$ and $y = 2 \sin \theta$, are shown together in Fig. 14-4. Each ordinate of the curve of $y = 2 \sin \theta$ is for any given value of θ twice the ordinate of the curve of $y = \sin \theta$. In particular, the maximum ordinate of $y = \sin 2\theta$ is 2. In general, a rotating vector of magnitude A generates by its projection the curve $y = A \sin \theta$ of maximum value A. The maximum value of a sine function is referred to as its *amplitude*.

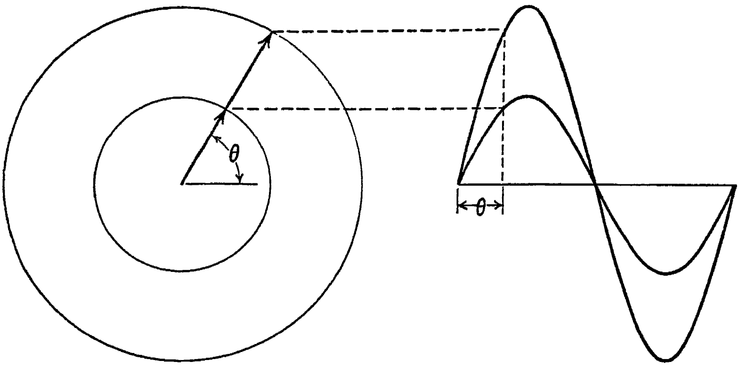


FIG. 14-4. Rotating vectors of lengths 1 and 2 generate $y = \sin \theta$ and $y = 2 \sin \theta$, respectively.

Exercise 14-2

$\mathbf{V}_1$ is a vector of magnitude 1. $\mathbf{V}_2$ is a vector of magnitude 2. $\mathbf{V}_2$ leads $\mathbf{V}_1$ by a constant phase difference of 45°. In Fig. 14-5 the sine wave projections associated with each of these rotating vectors are plotted against θ, the angle of $\mathbf{V}_1$; and the vectors themselves are shown in the position corresponding to $\theta = 0$. Construct similar diagrams for each of the following cases: (a) $\mathbf{V}_2$ leading $\mathbf{V}_1$ by 60°; (b) $\mathbf{V}_2$ lagging $\mathbf{V}_1$ by 30°.

14-4. Frequency and Period. For each revolution of a rotating vector the corresponding sine function completes one cycle. The number of cycles per second is called the *frequency*; and the number of seconds per

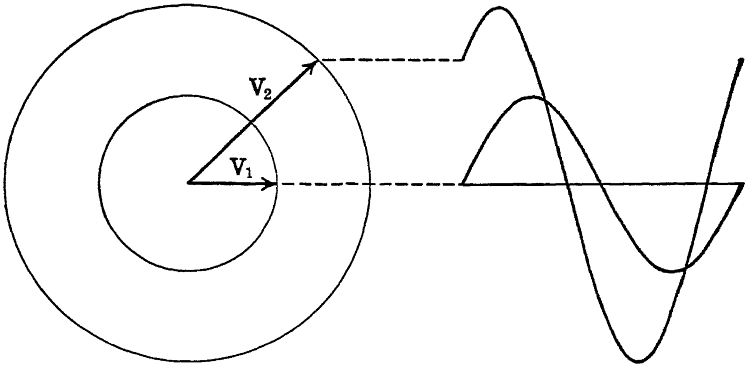


FIG. 14-5. Rotating vectors and curves of their projections; V_2 leading V_1 by 45°.

cycle is called the *period*. Frequency, f, is then the reciprocal of the period, T. For a vector which is rotating at a speed of 100 revolutions per second: (a) the frequency is 100 cycles per second; (b) the period is $\frac{1}{100}$ second per cycle; and (c) the angular velocity of the vector in radians per second is 100 times 2π, since there are 2π radians in each revolution. In general, the *angular velocity*, ω, is thus related to the frequency and to the period by the equation

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{14-1}$$

If t represents the time elapsed from the beginning of the rotation, then θ at any instant is given by

$$\theta = \omega t; \tag{14-2}$$

and the expression $y = A \sin \theta$ may be written $y = A \sin \omega t$.

Exercise 14-3

1. Given $\omega = 3$ radians per second, evaluate θ for the following times, t: 0, $\frac{1}{2}$, 10 seconds.

2. Express in radians per second: (a) 10 revolutions per second: (b) 1800 revolutions per minute.

3. What is the angular velocity of the rotating vector which generates a sine wave of frequency (a) 60 cycles per second; (b) 1,000,000 cycles per second?

14-5. Addition of Sine Functions. If the vectors **A**, magnitude A, and **B**, magnitude B, "photographed" in Fig. 14-6 are presumed to be

rotating at the same uniform speed, with **B** leading **A** by the constant angle ϕ, then the associated sine curves a and b are in amplitude A and B, respectively, with a constant phase difference ϕ between them. The

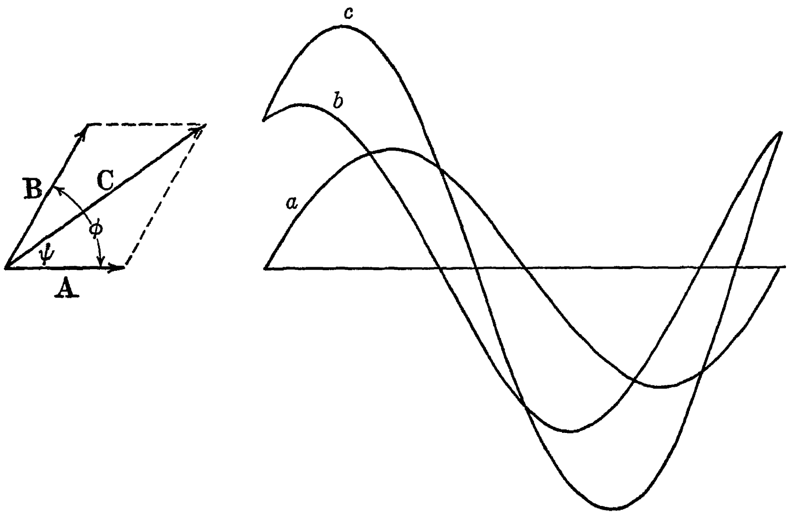


FIG. 14–6. Addition of sine functions.

equation of a is $y = A \sin \omega t$; the equation of b is $y = B \sin (\omega t + \phi)$. (Compare part F of Exercise 10–1.) C, the graph of the function $A \sin \omega t + B \sin (\omega t + \phi)$, is obtained through point-by-point addition of a and b. At the same time, however, the rotating vector **C**, which is the vector sum **A** and **B**, describes c through its (**C**'s) vertical projection. This is apparent from the fact that the vertical projection of **C** at each instant is equal to the sum of the vertical projections of **A** and **B**, as may readily be seen from the head-to-tail addition of Fig. 14–7. Since the curve c is thus associated with the rotating vector **C**, we conclude that c *is a sine curve of the same frequency as that of a and of b*; the amplitude of c is equal to the length of the vector **C**; with regard to phase c leads a by the constant angle ψ, the angle between **C** and **A**.

FIG. 14–7. Vertical projection of **C** at any instant equal to the sum of the vertical projections of **A** and **B**.

Exercise 14–4

1. Demonstrate that the equation of curve b in Fig. 14–6 is $y = B \sin(\omega t + \phi)$.

2. Reproduce the curves a and b of Fig. 14–6, and on the same diagram with the curves a and b draw that curve which is obtained by taking the point-by-point difference of the ordinates of a from the ordinates of b. Show that the function which is thus obtained from the difference of the two sine functions is itself a sine function and is represented on the rotating vector picture by $\mathbf{B} - \mathbf{A}$.

3. Consider the sum of two functions $y_1 = A_1 \sin \omega_1 t$ and $y_2 = A_2 \sin \omega_2 t$, where $A_2 = 2A_1$ and $\omega_2 = 2\omega_1$. Draw a series of graphs showing the representative vectors (both the individual vectors and the sum vector) at successive instants.

4. Given two rotating vectors which are equal in magnitude and in rotational speed, but opposite in direction of rotation; show that the sum of these vectors is a non-rotating vector which varies sinusoidally in magnitude.

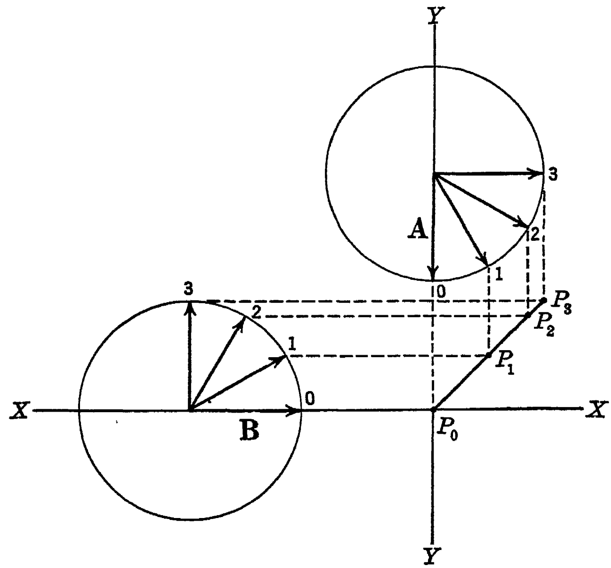


FIG. 14–8. Composition of projections. Generating vectors of same magnitude, of same angular velocity, and of same phase.

14–6. Composition of Projections. Lissajous' Figures. In Fig. 14–2 a sine curve is obtained as the trace of a point which undergoes simultaneously a motion to the right which is uniform at the rate $\theta = \omega t$, and

a motion up and down which follows the vertical projection of the rotating arrowhead. An interesting type of motion is that which ensues from the simultaneous application of sinusoidal projections both horizontally and vertically, as in Fig. 14–8. If the rotating vectors **A** and **B** are of the same magnitude, of the same angular velocity, and of the same phase, then the trace of the moving point P (the intersection of the projections) is a straight line at 45° to the axes. At the beginning of the motion, time $t = 0$, the magnitude of the horizontal projection of **B** is zero relative to a vertical axis YY through the center of the **B** circle; and the magnitude of the vertical projection of **A** is zero as measured relative to a horizontal axis XX through the center of the **A** circle. Then, at time $t = 0$, the coordinates of the point P relative to the axes XX and YY are (0,0). This situation is represented by P_0 in Fig. 14–8. When the **A** vector advances through 30° to position 1, its vertical projection becomes A sin 30°; the **B** vector, progressing at the same rate, simultaneously reaches its position 1 with a horizontal projection of B sin 30°. The coordinates of the point P relative to axes XX and YY are then: abscissa, B sin 30°; ordinate, A sin 30°. This situation is represented by P_1. Only one quarter of the cycle is shown in Fig. 14–8. During the second quarter-cycle the moving point P retraces its path, and during the third quarter-cycle it extends downward, returning to its original position, P_0, at the completion of the cycle.

Fig. 14–9 shows the pattern traced by the intersection of the projections when the generating vectors, **A** and **B**, are of the same angular velocity but with **A** of twice the magnitude of **B**. In Fig. 14–10 **A** and **B** are of the same magnitude and angular velocity, but **A** leads **B** in phase by 90°. In Fig. 14–11, and Fig. 14–12 the angular velocity of **A** is twice that of **B**. In Fig. 14–11, **A** and **B** are initially in phase, whereas in Fig. 14–12, **A** initially leads **B** by 90°. In each case in Fig. 14–11 and Fig. 14–12, since the velocities of **A** and **B** differ, the phase difference between the two vectors varies as the vectors rotate. Fig. 14–13 illustrates the patterns traced by the intersection of the projections of the rotating vectors of two *cosine* functions of equal magnitude under various circumstances of angular velocity and initial phase.

Designs obtained by the composition of two periodic motions in the manner described in this section are known as *Lissajous' figures*. In any particular pattern let us refer to a point of maximum ordinate as a *peak* and a point of maximum abscissa as an *end loop*. With this notation the trace of Fig. 14–11 has two peaks and one end loop. With reference to

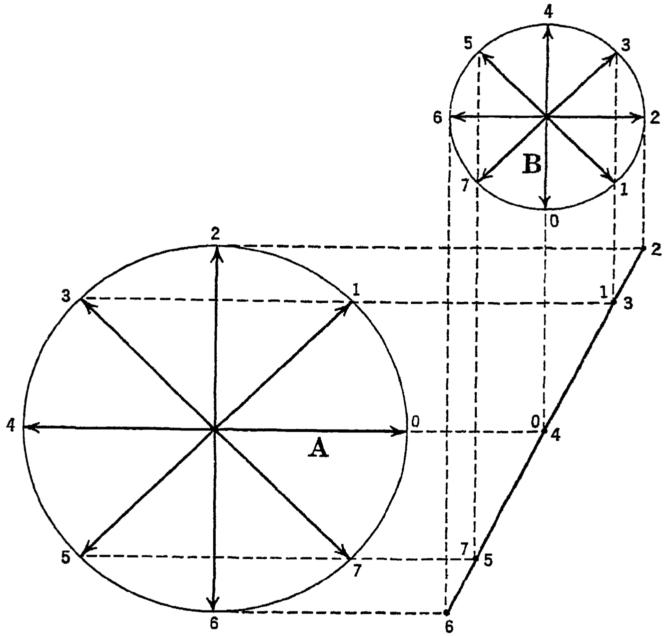


FIG. 14-9. Magnitude of A twice that of B.

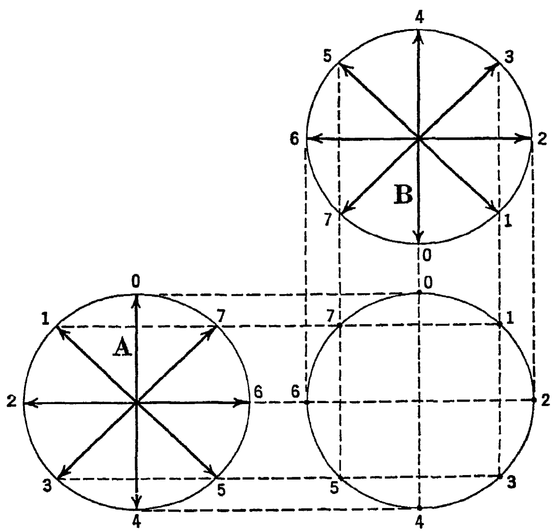


FIG. 14-10. A leading B in phase by 90°.

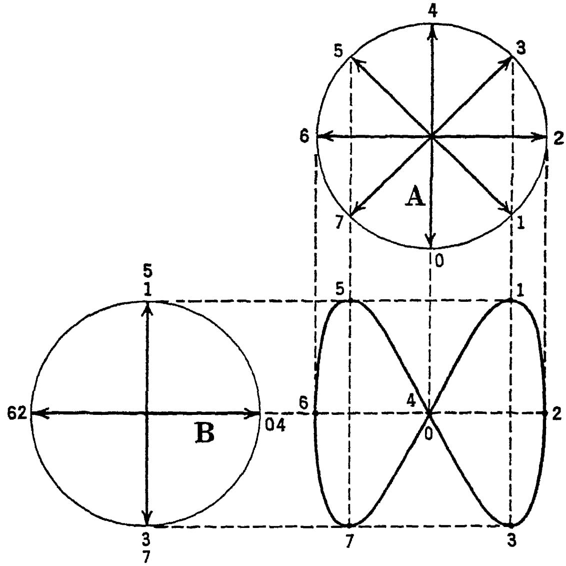


FIG. 14-11. Angular velocity of **A** twice that of **B**.

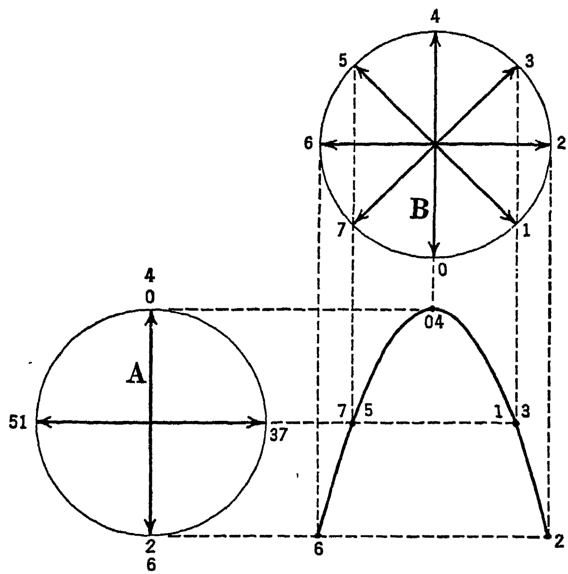


FIG. 14-12. Angular velocity of **A** twice that of **B**; **A** initially leading **B** in phase by 90°.

Fig. 14-13 we see that the outside patterns in each row are special in that in each outside pattern the moving point retraces its path over a portion of the cycle so that parts of the trace coincide. Except for these special cases the relative number of peaks and end loops in a Lissajous pattern is the same as the ratio of the velocities (or of the frequencies) of the two generating vectors. In the center three patterns of the

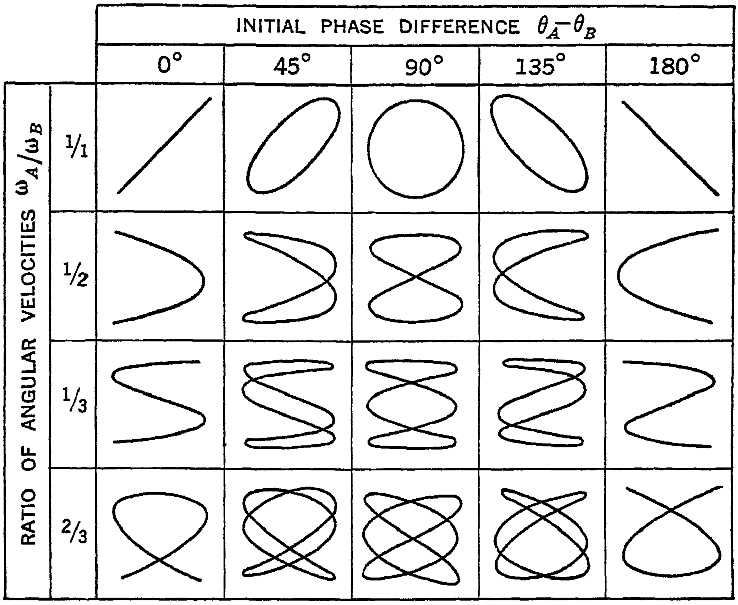


Fig. 14-13. Composition of projections under various circumstances for two cosine functions.

second row of Fig. 14-13 there are one peak and two end loops; here the velocity ratio of the generating vectors is 1 to 2. In the center three patterns of the third row of Fig. 14-13 there are one peak and three end loops corresponding to a velocity ratio of the generating vectors of 1 to 3. And in the center three patterns of the fourth row of Fig. 14-13 there are two peaks and three end loops corresponding to a velocity ratio of the generating vectors of 2 to 3.

Exercise 14–5

Develop Lissajous' figures for pairs of generating vectors **A** and **B** as indicated in the following table:

	Magnitude	Angular Velocity	Initial Phase
1.	$\|\mathbf{A}\| = \|\mathbf{B}\|$	$\omega_A = \omega_B$	$\theta_A = 30°;\ \theta_B = 0°$
2.	$\|\mathbf{A}\| = \|\mathbf{B}\|$	$\omega_A = \omega_B$	$\theta_A = 0°;\ \theta_B = 30°$
3.	$\|\mathbf{A}\| = \|\mathbf{B}\|$	$\omega_A = 2\omega_B$	$\theta_A = 90°;\ \theta_B = 0°$
4.	$2\|\mathbf{A}\| = \|\mathbf{B}\|$	$\omega_A = 2\omega_B$	$\theta_A = 0°;\ \theta_B = 0°$
5.	$2\|\mathbf{A}\| = 3\|\mathbf{B}\|$	$3\omega_A = 2\omega_B$	$\theta_A = 0°;\ \theta_B = 0°$

Exercise 14–6

1. Modulation of a radiated signal to convey intelligence is accomplished in a frequency-modulated transmitter by varying the phase of the transmitted signal above and below its normal (unmodulated) value. The receiver is designed to produce an output response only in accordance with such phase variations. In Fig. 14–14 the voltage induced in the receiving antenna by the desired signal is represented at some particular instant by **D**, and the voltage induced simultaneously by an undesired signal of approximately the same frequency (which cannot be rejected by the receiver's selective circuits) is represented by **U**. The resultant of these two voltages together is represented by **R**. The receiver does not respond to amplitude variations, so that the increased amplitude caused by the addition of **U** to **D** is of no concern. However, the phase shift ϕ introduces distortion.

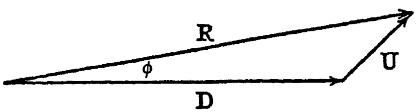


FIG. 14–14. Vector representation of the effect of interference in a frequency modulation receiver.

Show that, if $|\mathbf{U}|$ is less than $|\mathbf{D}|$, the amount of such an undesired phase shift is at worst given by

$$\phi = \sin^{-1}\frac{|\mathbf{U}|}{|\mathbf{D}|}.$$

so that the percentage distortion produced by an interfering signal can be reduced by employing intentionally wide variations in the phase of the desired signal during modulation at the transmitter. Whereas the distortion phase

shift cannot exceed $\sin^{-1} \frac{|\mathbf{U}|}{|\mathbf{D}|}$, the intentional intelligence phase shift produced at the transmitter may be of the order of 5 to 500 radians, thus minimizing the relative effect of interference.

2. In a cathode ray oscilloscope the extent of the horizontal deviation of the fluorescent spot from the central position on the screen is proportional to the voltage applied across a pair of horizontally deflecting plates within the tube, while the extent of the vertical deviation of the spot from the center is proportional to the voltage applied across a pair of vertically deflecting plates within the tube. Determine the nature of the pattern traced by the fluorescent spot on the screen if the same 60-cycle voltage is applied simultaneously across both pairs of plates. How would the observed pattern appear if the frequency of the applied voltage was increased to 500 cycles per second?

3. The oscilloscope may be used for frequency calibration. Suppose the pattern of Fig. 14–15 appears on the screen of an oscilloscope when a 1000-cycle sinusoidal voltage is applied to the vertically deflecting plates and sinusoidal voltage of unknown frequency is simultaneously applied to the horizontally deflecting plates. Determine the frequency of the voltage which is applied to the horizontally deflecting plates.

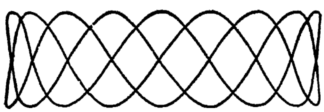


FIG. 14–15. Frequency determination by means of oscilloscope.

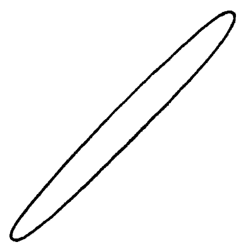


FIG. 14–16. Phase shift determination by means of oscilloscope.

4. To measure the phase shift introduced by an amplifier, a sinusoidal voltage is applied both to the amplifier input terminals and to the horizontally deflecting plates of an oscilloscope. The amplifier output voltage (same frequency and same wave form as the input voltage) is applied simultaneously to the vertically deflecting plates of the oscilloscope with a resultant pattern as shown in Fig. 14–16. Determine the phase shift introduced by the amplifier. (From the dimensions of the oscilloscope pattern determine the radii of each of the rotating vectors. Draw the circles which represent the loci of the rotating vectors. Select any point on the oscilloscope pattern, and for this particular point correlate the positions of the two rotating vectors.)

5. In order that an amplifier preserve wave form it is necessary that the amplifier introduce either (a) no phase shift, or (b) a phase shift which is proportional to the frequency. Prove (b) for the case of an input signal of the form

$$e_g = E_1 \sin (\omega t) + E_2 \sin (2\omega t) + E_3 \sin (3\omega t).$$

(Assuming that voltage amplification, A, is constant for all frequencies, but that phase shift is proportional to the frequency, the output signal is of the form

$$e_p = AE_1 \sin (\omega t + \phi) + AE_2 \sin (2\omega t + 2\phi) + AE_3 \sin (3\omega t + 3\phi).$$

Compare the shapes of the curves of e_g and e_p.)

CHAPTER 15

VECTOR FORMS

15–1. Polar and Rectangular Forms. In Chapter 13 we described vectors by magnitude and angle. An alternative scheme of describing vectors is in terms of horizontal and vertical components. The vector $4\underline{/150^\circ}$ of Fig. 15–1 may be described as having a horizontal component equal to -2.87 and a vertical component equal to 2. The first form of describing a vector, by magnitude and angle, is called the *polar form.* The second form of describing a vector, by horizontal and vertical components, is called the *rectangular form.*

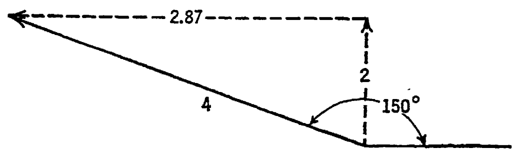


FIG. 15–1. Vector $4\underline{/150^\circ}$.

In listing components in the rectangular form it is customary to write horizontal and vertical components, in that order, with a letter j prefixing the vertical component. Thus: $4\underline{/150^\circ} = -2.87 + j2$; or, in general, $M\underline{/\theta} = x + jy$, where $M = x^2 + y^2$ and $\theta = \tan^{-1}\dfrac{y}{x}\cdot$ The practice of writing $x + jy$ to denote the horizontal and vertical components of a vector resembles the practice of writing (x,y) to denote the horizontal and vertical coordinates of a point, the only essential difference being in the manner of designating the vertical element, that is, by a letter j instead of by a separating comma.

A vector of the type $A + j1$ may be written as $A + j$, the quantity 1 being understood. A vector of the type of $A + j0$ may be written as simply A, and a vector of the type $0 + jB$ may be written as simply jB.

Writing $A + j0$ as A introduces a possible ambiguity in that the quantity A might represent either a vector of angle zero or a scalar (of no angle). However, the notation rarely causes any confusion since in any writing the context usually indicates which interpretation is intended.

Exercise 15–1

A. Express each of the following vectors in the rectangular form:

1. $2\,\underline{/45^\circ}$.
2. $5\,\underline{/120^\circ}$.
3. $4\,\underline{/0^\circ}$.
4. $10\,\underline{/270^\circ}$.
5. $2\,\underline{/225^\circ}$.
6. $15\,\underline{/270^\circ}$. (Here the y-component is negative. $15\,\underline{/270^\circ} = 0 - j15 = -j15$.)
7. $1\,\underline{/90^\circ}$.
8. $3\,\underline{/-30^\circ}$.

B. Perform the following indicated multiplications wherein in each case the number which is not in parentheses is a scalar. Express the results in rectangular form.

1. $2(j)$.
2. $2(3 + j)$.
3. $5(1 - j4)$.
4. $-5(1 - j4)$.
5. $-\frac{1}{2}(-2 + j3)$.
6. $\dfrac{1}{A}(L + jM)$.

15–2. Vector Addition. In Fig. 15–2 the vector **C** is the sum of the vectors **A** and **B**. It is apparent from the figure that the horizontal component of **C** is the sum of horizontal components of **A** and **B**, and that the vertical component of **C** is the sum of the vertical components of **A** and **B**. Analytical addition of two vectors which are in the rectangular form is, thus, a simple matter.

In Fig. 15–3 are shown the vectors $2 + j3$ and $5 - j7$. The sum of these vectors is

$$(2 + 5) + j(3 - 7) = 7 - j4.$$

Exercise 15–2

Perform analytically the following indicated vector operations. (First express each vector in the rectangular form if it is not already in that form.) Check the result in each case with a graphical solution.

1. $(1 + j2) + (5 + j4)$.
2. $(-2 + j6) + (3 - j4)$.
3. $(j2) + (-3)$.
4. $2\,\underline{/30^\circ} + 2\,\underline{/60^\circ}$.
5. $2\,\underline{/30^\circ} + 2\,\underline{/120^\circ}$.
6. $3\,\underline{/17^\circ} + 5\,\underline{/268^\circ}$.
7. $3\,\underline{/17^\circ} + (4 - j2)$.
8. $(-1 + j8) + (j5) + (12 - j2)$.

15–3. Vector Subtraction. Fig. 15–2 may be regarded as representing the vector relation: $\mathbf{C} - \mathbf{A} = \mathbf{B}$. From this viewpoint of Fig. 15–2 it is apparent that, for two vectors which are in the rectangular form, subtraction of the vectors is accomplished by the subtraction of their components. For example, (Fig. 15–3): $(7 - j4) - (2 + j3) = (7 - 2) + j(-4 - 3) = 5 - j7$.

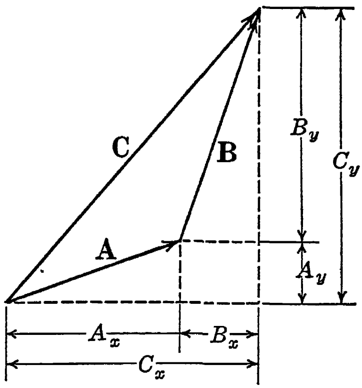


FIG. 15–2. $\mathbf{A} + \mathbf{B} = \mathbf{C}$. The sum of the components of **A** and **B** along an axis is equal to the component of **C** along that axis.

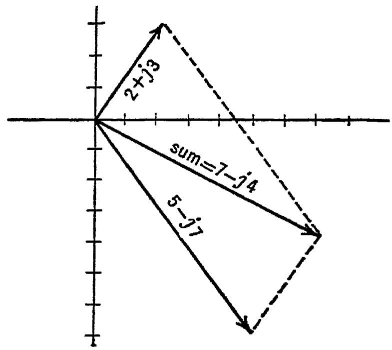


FIG. 15–3. $(2 + j3) + (5 - j7)$.

Exercise 15–3

Perform analytically the following indicated vector operations. (Compare Exercise 15–2.)

1. $(1 + j2) - (5 + j4)$.
2. $(-2 + j6) - (3 - j4)$.
3. $(j2) - (-3)$.
4. $2\,\underline{/30^\circ} - 2\,\underline{/60^\circ}$.
5. $2\,\underline{/30^\circ} - 2\,\underline{/120^\circ}$.
6. $3\,\underline{/17^\circ} - 5\,\underline{/268^\circ}$.
7. $3\,\underline{/17^\circ} - (4 - j2)$.

15–4. Vector Multiplication. We are at liberty to define what we shall call the *product of one vector by another vector*, so long as the concept of vector product is logically compatible with the established concept of vector addition (and with the related notions of vector subtraction, and of multiplication and division of a vector by a scalar). We choose the following definition of vector multiplication here, because we find for

our purposes — in the study of alternating-current circuits — that it leads to a consistent graphical portrayal, in certain useful cases, of what we mean by the product of two physical quantities. We define the product of two vectors as a third vector: (a) the magnitude of which is equal to the product of the magnitudes of the factor vectors, and (b) the angle of which is equal to the sum of the angles of the factor vectors, or equal to any angle which is equivalent to this sum (coterminal angle). The product of the vectors $2\,\underline{/10^\circ}$ and $3\,\underline{/20^\circ}$ is, then, according to our definition

$$(2 \cdot 3)\,\underline{/10^\circ + 20^\circ} = 6\,\underline{/30^\circ};$$

and the product of the vectors $5\,\underline{/20^\circ}$ and $3\,\underline{/-12^\circ}$ is

$$(5 \cdot 3)\,\underline{/20^\circ - 12^\circ} = 15\,\underline{/8^\circ}.$$

It follows from the definitions of vector addition and vector multiplication that operations with vectors obey the same commutative, associative, and distributive laws as do operations with ordinary numbers (Exercise 15–4B).

Exercise 15–4

A. Perform the following indicated vector operations:

1. $(2\,\underline{/30^\circ}) \cdot (2\,\underline{/60^\circ})$.
2. $(1\,\underline{/45^\circ}) \cdot (3\,\underline{/-60^\circ})$.
3. $(3 + j4) \cdot (-5 + j12)$. (First convert each vector to the polar form.)

B. Assuming the laws of operations for ordinary numbers, namely, Eqs. (5–1) through (5–5), show that vector operations follow these same laws.

15–5. Vector Division. We define *vector division* as the inverse of vector multiplication. If $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$, we say that $\dfrac{\mathbf{C}}{\mathbf{B}} = A$. The problem $\dfrac{\mathbf{C}}{\mathbf{B}} = ?$ is then the same as the problem $? \cdot \mathbf{B} = \mathbf{C}$. The unknown quantity indicated by the question mark is by the definition of vector product a vector: (a) whose magnitude when multiplied by the magnitude of **B** equals the magnitude of **C**, and (b) whose angle when added to the angle of **B** equals the angle of **C**. In other words, the magnitude of the desired vector is the quotient of the magnitudes of **C** and **B**, and the angle of the desired vector is the difference of the angles of **C** and **B**. If $\mathbf{C} = 2/10^\circ$

and if $\mathbf{B} = 4\underline{/70^\circ}$, then

$$\frac{\mathbf{C}}{\mathbf{B}} = \frac{2\underline{/10^\circ}}{4\underline{/70^\circ}} = \frac{2}{4}\underline{/10^\circ - 70^\circ} = 0.5\underline{/-60^\circ}.$$

Exercise 15-5

A. Perform the following indicated vector operations:

1. $\dfrac{8\underline{/45^\circ}}{2\underline{/3^\circ}}$.
3. $\dfrac{40\underline{/22^\circ}}{3 + j4}$.
5. $\sqrt{-1}$.
7. $\sqrt[3]{-8}$.
2. $\dfrac{3\underline{/-7^\circ}}{5\underline{/2^\circ}}$.
4. $\sqrt{16\underline{/30^\circ}}$.
6. $\sqrt{-3}$.
8. $-2^{\frac{3}{2}}$.

B. Show that

$$\frac{\mathbf{A} \cdot \mathbf{C}}{\mathbf{B} \cdot \mathbf{D}} = \frac{\mathbf{A}}{\mathbf{B}} \cdot \frac{\mathbf{C}}{\mathbf{D}},$$

where **A, B, C,** and **D** are any vectors. (The vector expression $\dfrac{\mathbf{A} \cdot \mathbf{C}}{\mathbf{B} \cdot \mathbf{D}}$ resembles in form the algebraic expression $\dfrac{A \cdot C}{B \cdot D}$, but it does not follow from this similarity in appearance of the posed operations that the results of the vector and algebraic operations should likewise be similar in appearance. We have defined the product of two vectors as a third vector which has a magnitude equal to the product of the magnitudes of the two given vectors, and an angle equal to the sum of the angles of the two given vectors. Considering this definition and the definition of vector division, it would be naive to suppose that no proof is required for the statement

$$\frac{\mathbf{A} \cdot \mathbf{C}}{\mathbf{B} \cdot \mathbf{D}} = \frac{\mathbf{A}}{\mathbf{B}} \cdot \frac{\mathbf{C}}{\mathbf{D}}.$$

Suggestion for proof: Write

$$\frac{\mathbf{A}}{\mathbf{B}} = \mathbf{M}, \text{ and } \frac{\mathbf{C}}{\mathbf{D}} = \mathbf{N},$$

so that $\quad \mathbf{A} = \mathbf{B} \cdot \mathbf{M}$ and $\mathbf{C} = \mathbf{D} \cdot \mathbf{N}$.)

C. Using the result of Problem *B* above, show that

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{A} \cdot \mathbf{C}}{\mathbf{B} \cdot \mathbf{C}}$$

where **A, B,** and **C** are any vectors.

15-6. Product of Sine Functions. It might be assumed that the product of two rotating vectors, taken according to the definition of vector product in Sec. 15-4, is equivalent to point-by-point multiplication of the corresponding sine function in analogous manner to the vector and point-by-point addition discussed in Sec. 14-5. That this is not the case may be demonstrated in several ways. Not the most elegant but perhaps the simplest way to disprove any general proposition is to cite one instance where the proposition fails. In the present situation it is adequate to observe in Fig. 14-6 that at the time $t = 0$ the product of **A** and **B** is a vector of magnitude $A \cdot B$ in the direction of B, and the vertical projection of this product is definitely not zero; whereas the product of the vertical projections is zero at time $t = 0$, since the projection of **A** is zero at time $t = 0$. The vector product, as defined in Sec. 14-5, nevertheless, serves a useful purpose in electrical circuit analysis in the representation of phenomena involving the product of two vectors of which at least one is non-rotating. An example of a non-rotating vector is a vector that represents an impedance which is constant in time. An example of a rotating vector is a vector that represents through its vertical projection a potential which is sinusoidally varying with time.

15-7. Special Simple Vector Products and Quotients. Since a vector jA expressed in the polar form is $A\underline{/90°}$, it follows that the vector product $(jA) \cdot (B\underline{/\theta})$, where $B\underline{/\theta}$ represents any given vector, is a vector $AB\underline{/\theta + 90°}$. And since $A + j0$ expressed in the polar form is $A\underline{/0°}$, it follows that the vector product $(A + j0) \cdot (B\underline{/\theta})$ is equal to $AB\underline{/\theta}$.

Further,

$$\frac{B\underline{/\theta}}{jA} = \frac{B\underline{/\theta}}{A\underline{/90°}} = \frac{B}{A}\underline{/\theta - 90°},$$

and

$$\frac{B\underline{/\theta}}{(A + j0)} = \frac{B\underline{/\theta}}{A\underline{/0°}} = \frac{B}{A}\underline{/\theta}.$$

If in the rectangular form $B\underline{/\theta}$ is given by $L + jM$, then $\frac{B}{A}\underline{/\theta}$ in the rectangular form is given by $\frac{L}{A} + j\frac{M}{A}$. And the equation

$$\frac{B\underline{/\theta}}{A\underline{/0°}} = \frac{B}{A}\underline{/\theta}$$

may be written as

$$\frac{L + jM}{A + j0} = \frac{L}{A} + j\,\frac{M}{A}.$$

Exercise 15–6

In the following problems a number standing by itself represents a vector of the type $A + j0$, and not a scalar quantity. Perform the indicated vector operations:

1. $(j) \cdot (3\underline{/25°})$.
2. $(2j) \cdot (3)$. (Note: $3 = 3 + j0$.)
3. $(-2j) \cdot (12\underline{/-10°})$. (Note: $-2j = 0 - 2j = 2\underline{/270°}$.)
4. $(6) \cdot (-j3)$.
5. $(-6) \cdot (j3)$.
6. $(2) \cdot (5)$.
7. $(j) \cdot (j)$.
8. $(j) \cdot (-j)$.
9. $(j4) \cdot (j7)$.
10. $\dfrac{12 - j9}{3}$.

15–8. The Operator j. Inasmuch as any vector, $M\underline{/\theta}$, when multiplied by j yields $M\underline{/\theta + 90°}$, we may if we wish, regard j as an *operator* which is such that when applied to any given vector, it serves to rotate that vector through an angle of 90°. This viewpoint of j as an operator is frequently adopted by engineering texts.

15–9. Conjugate Vectors. The conjugate of any given vector is the mirror image of that vector about the x-axis (Fig. 15–4). The conjugate of a vector $A\underline{/\theta}$ is $A\underline{/-\theta}$. And the conjugate $A + jB$ is $A - jB$. The product of any two conjugate vectors is a vector which is of zero angle and which is in magnitude equal to the square of the magnitude of one of the conjugate vectors. For example,

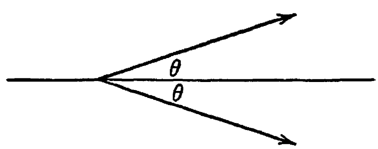


FIG. 15–4. Conjugate vectors.

$$3\underline{/10°} \cdot 3\underline{/-10°} = 9\underline{/0°}, \text{ and } (3 + j4) \cdot (3 - j4) = 25 + j0 = 25.$$

Exercise 15–7

Perform the indicated vector operations. In each case draw the given vectors and the product vector.

1. $6\underline{/42^\circ} \cdot 6\underline{/-42^\circ}$. **2.** $(-12 + j5) \cdot (-12 - j5)$.

15–10. Multiplication of Vectors in the Rectangular Form. Two vectors which are in the rectangular form may be multiplied term by term in the manner of multiplying two binomials. The justification of the method lies in the fact that the resulting vector satisfies the definition of vector product. Term by term multiplication of $A + jB$ by $C + jD$ yields:

$$(A + jB) \cdot (C + jD) = A \cdot C + A \cdot jD + jB \cdot C + jB \cdot jD.$$

Each of the four products on the right side of this equation may be expressed as a single vector. Thus:

$$A \cdot C = AC;\, A \cdot jD = jAD;\, jB \cdot C = jBC; \text{ and } jB \cdot jD = -BD;$$

so that

$$\begin{aligned}(A + jB) \cdot (C + jD) &= AC + jAD + jBC - BD\\ &= (AC - BD) + j(AD + BC).\end{aligned}$$

To demonstrate that the resulting vector, $(AC - BD) + j(AD + BC)$, is actually the vector product of $A + jB$ and $C + jD$ it is necessary to show: (1) that the magnitude of $(AC - BD) + j(AD + BC)$ is equal to the product of the magnitudes of $A + jB$ and $C + jD$, and (2) that the angle of $(AC - BD) + j(AD + BC)$ is equal to the sum of the angles of $A + jB$ and $C + jD$ (or equal to an angle which is equivalent to this sum). The proof of the magnitude relation amounts to demonstrating that

$$\sqrt{(AC - BD)^2 + (AD + BC)^2} = \sqrt{A^2 + B^2} \cdot \sqrt{C^2 + D^2}. \quad (15\text{–}1)$$

The details of this proof are left as an exercise (Problem A, Exercise 15–8). The proof of the angle relation amounts to demonstrating that if θ and ϕ are the angles of the given vectors and if ψ is the angle of the resulting vector, then θ, ϕ, and ψ satisfy both the equations

$$\sin\psi = \sin\theta\cos\phi + \cos\theta\sin\phi \quad (15\text{–}2)$$

and

$$\cos\psi = \cos\theta\cos\phi - \sin\theta\sin\phi. \quad (15\text{–}3)$$

[Compare Eqs. (9–13) and (9–14).] Eqs. (15–2) and (15–3) will be shown in Sec. 19–2 to constitute a sufficient condition for ψ to be the sum of θ and ϕ (or an angle equivalent to this sum). It is left as an exercise for the student (Problem B of Exercise 15–8) to demonstrate that the angles of the three vectors in question satisfy Eqs. (15–2) and (15–3). For the present it may be stated that both of the requirements of the product vector are satisfied by the vector $(AC - BD) + j(AD + BC)$ and, hence, that the method of term-by-term multiplication illustrated above is valid.

For a practical application of the method let us multiply $3 + j4$ by $-5 + j2$. In Problem A 3 of Exercise 15–4 this same product was required to be obtained first by converting each vector to the polar form.

$$\begin{aligned}(3 + j4) \cdot (-5 + j2) &= 3 \cdot (-5) + 3 \cdot j2 + j4 \cdot (-5) + j4 \cdot j2 \\ &= -15 + j6 - j20 - 8 \\ &= -23 - j14.\end{aligned}$$

In the polar form $-23 - j14$ is $26.9\underline{/211^\circ\ 40'}$. This is the same value as that obtained in Problem A 3 of Exercise 15–4.

Exercise 15–8

A. Prove Eq. (15–1). [On the left side of Eq. (15–1) expand $(AC - BD)^2$ and $(AD + BC)^2$ to obtain, respectively, $A^2C^2 - 2ABCD + B^2D^2$ and $A^2D^2 + 2ABCD + B^2C^2$. Express the right side of Eq. (15–1) as

$$\sqrt{(A^2 + B^2) \cdot (C^2 + D^2)} = \sqrt{A^2C^2 + B^2C^2 + A^2D^2 + B^2D^2}.]$$

B. Show that if:

$$A + jB = M\underline{/\theta},$$
$$C + jD = N\underline{/\phi},$$

and

$$(AC - BD) + j(AD + BC) = P\underline{/\psi},$$

then θ, ϕ and ψ satisfy both Eqs. (15–2) and (15–3). [With reference to Fig. 15–5

$$\sin\theta = \frac{B}{M}, \quad \cos\theta = \frac{A}{M}$$

$$\sin\phi = \frac{D}{N}, \quad \cos\phi = \frac{C}{N};$$

$$\sin\psi = \frac{AD + BC}{P}, \quad \cos\psi = \frac{AC - BD}{P}.$$

Thus,

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{B}{M}\cdot\frac{C}{N} + \frac{A}{M}\cdot\frac{D}{N},$$

which becomes, on making use of the relation $MN = P$, (which was proven in Problem A above),

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{BC}{P} + \frac{AD}{P}.\Big]$$

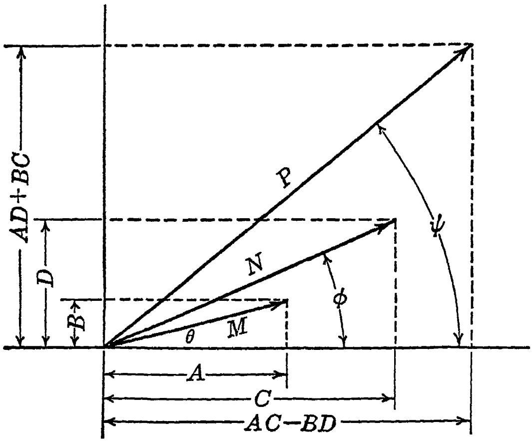


Fig. 15-5. Vector multiplication: $(A + jB)\cdot(C + jD) = (AC - BD) + j(AD + BC)$.

C. Perform the indicated vector operations. Give the results in the rectangular form.

1. $(2 + j3)\cdot(4 + j5)$.
2. $(-1 + j2)\cdot(3 - j7)$.
3. $(5 - j)\cdot(-2 + j4)$.

15-11. Division of Vectors in the Rectangular Form. Division of vectors in the rectangular form may be accomplished as illustrated by the following:

$$\frac{A + jB}{C + jD} = \frac{(A + jB)\cdot(C - jD)}{(C + jD)\cdot(C - jD)}$$ (See Problem C of Exercise 15-5).

$$= \frac{AC - jAD + jBC + BD}{C^2 + D^2}$$ on multiplying out numerator and denominator,

$$= \frac{(AC + BD) + j(BC - AD)}{C^2 + D^2}$$

$$= \frac{AC + BD}{C^2 + D^2} + j\frac{BC - AD}{C^2 + D^2}$$

in accordance with the results of the last paragraph of Sec. 15-7 (on noting that $C^2 + D^2$ is a vector of angle zero).

Example. Divide $2 + j3$ by $4 + j5$.

$$\frac{2 + j3}{4 + j5} = \frac{(2 + j3) \cdot (4 - j5)}{(4 + j5) \cdot (4 - j5)}$$

$$= \frac{8 + j12 - j10 + 15}{4^2 + 5^2}$$

$$= \frac{(8 + 15) + j(12 - 10)}{16 + 25}$$

$$= \frac{23 + j2}{41}$$

$$= \frac{23}{41} + j\frac{2}{41}.$$

Exercise 15-9

Perform the indicated vector operations. Give the results in the rectangular form.

1. $\dfrac{3 + j2}{5 + j4}$.
2. $\dfrac{-1 + j2}{3 - j7}$.
3. $\dfrac{5 - j}{-2 + j4}$.

15-12. Complex Numbers. A quantity as $A + jB$ may be considered abstractly without regard to the geometric manner in which it was defined in Sec. 15-1. From the abstract point of view $A + jB$ is simply a number pair (A and B) or, if we wish, a new type of number. As a new type of number it forms a logical extension of our number system since it behaves in accordance with the five basic rules of ordinary number operations, [Eqs. (5-1) through (5-5)] and, further, when B is zero, its properties are identical with those of ordinary numbers. When $A + jB = M\underline{/\theta}$ is regarded thus abstractly, it is spoken of as a *complex number*. A is called its *real part*, and B is called its *imaginary part*. M is called its *modulus*, and θ is called its *amplitude*.

For any equation of the type $Ax^2 + Bx + C = 0$, wherein A, B, and

C are real quantities, the solutions each can be expressed by a complex number. For example, the solutions of $x^2 + x + 1 = 0$ are

$$x = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$$

and

$$x = -\frac{1}{2} - \frac{\sqrt{-3}}{2}.$$

Sometimes in communications work, with an equation of the type $Ax^2 + Bx + C = 0$, a solution which involves an imaginary quantity signifies that the condition imposed by the given equation is incompatible with the natural course of the phenomena involved.

The foregoing does not imply that we always associate imaginary quantities only with unrealities. The vector interpretation of a complex number alone testifies adequately to the practicability of describing realistic quantities with imaginary numbers.

In pure mathematics the letter i replaces our letter j in the treatment of complex numbers. For electrical work the letter i is reserved to represent current.

15-13. Cjs and Exponential Forms. The magnitude, M, and the angle, θ, of a vector $A + jB$ are given through the relations (see, for example, Fig. 15-5):

$$\frac{A}{M} = \cos\theta,$$

$$\frac{B}{M} = \sin\theta,$$

and

$$M = \sqrt{A^2 + B^2}.$$

Thus, we have

$$A = M\cos\theta,$$

$$B = M\sin\theta,$$

and

$$\begin{aligned} A + jB &= M\cos\theta + jM\sin\theta \\ &= M(\cos\theta + j\sin\theta). \end{aligned} \tag{15-4}$$

$M(\cos\theta + j\sin\theta)$ is sometimes written in abbreviated form as $M \text{ cjs } \theta$, and a vector expressed in this manner is said to be in the *cjs form*.

The additive angle feature of the vector product together with the multiplicative magnitude relation suggests that we might define an expression $\epsilon^{j\theta}$ which is such that

$$\epsilon^{j\theta} = 1\underline{/\theta}. \qquad (15\text{–}5)$$

Then in place of $M\underline{/\theta}$ we may write $M\epsilon^{j\theta}$; in place of $N\underline{/\phi}$ we may write $N\epsilon^{j\phi}$; in place of $MN\underline{/\theta + \phi}$ we may write $MN\epsilon^{j(\theta+\phi)}$; and in place of $MN\underline{/\theta - \phi}$ we may write $MN\epsilon^{j(\theta-\phi)}$. With this understanding,

$$(M\underline{/\theta}) \cdot (N\underline{/\theta}) = MN\underline{/\theta + \phi}$$

becomes

$$M\epsilon^{j\theta} \cdot N\epsilon^{j\phi} = MN\epsilon^{j(\theta+\phi)}; \qquad (15\text{–}6)$$

and

$$\frac{M\underline{/\theta}}{N\underline{/\phi}} = \frac{M}{N}\underline{/\theta - \phi}$$

becomes

$$\frac{M\epsilon^{j\theta}}{N\epsilon^{j\phi}} = \frac{M}{N}\epsilon^{j(\theta-\phi)}. \qquad (15\text{–}7)$$

Such an expression as $\epsilon^{j\theta}$ in itself is pure formalism since our dealings with exponents do not extend to vector or operator exponents. However, it is apparent from Eqs. (15–6) and (15–7) that the behavior of $\epsilon^{j\theta}$ is consistent with the rules for expressions involving ordinary exponents; and for the present this justifies our choice of symbolism. It can be shown in the study of the complex variable that ϵ as defined from $\epsilon^{j\theta}$ in Eq. (15–5) is identical with $\epsilon = 2.718\ldots$ ordinarily defined by the limit of the quantity $\left(1 + \frac{1}{m}\right)^m$ as m increases indefinitely. (Compare Sec. 6–5.)

A vector in the form $M\epsilon^{j\theta}$ is said to be in the *exponential form.* $\epsilon^{j\theta}$ is sometimes written $\exp j\theta$ (read: "exponential $j\theta$"). The notation $\exp j\theta$ is favored over $\epsilon^{j\theta}$ in general printed work because of the ease with which $\exp j\theta$ can be set up in type as compared with $\epsilon^{j\theta}$.

Exercise 15–10

A. Express each of the following vectors (a) in the cjs form, (b) in the exponential form:

1. $2 + j3$.
2. $-5 - j$.

B. Express each of the following vectors in the rectangular form:

1. $2\epsilon^{j45°}$.
2. 5 cjs 350°.
3. $5\epsilon^{j(-10°)}$.
4. -5 cjs 170°.

C. Show that the conjugate of $M\epsilon^{j\theta}$ is $M\epsilon^{-j\theta}$.

15-14. Survey of Vector Forms for Computations. The polar, cjs, and exponential forms are similar in their manner of handling in vector computations, and in the following remarks they are all included in the category of the polar form. For vector addition or subtraction it is essential that the vectors be in the rectangular form. For multiplication or division either the rectangular or the polar form is satisfactory. The choice of which form to use for multiplication or division is best based on considerations of: (a) in what form the vectors are given and (b) in what form the results are desired. Multiplying or dividing vectors in the rectangular form is usually less trouble than converting to the polar form, then multiplying or dividing, and then converting back again. However, for the multiplication or division process alone the work is simplified if the vectors are in the polar form.

15-15. Similarity of Indicated Vector and Scalar Operations. Algebraic operations with vectors in any form follow the same commutative, associative, and distributive laws as do algebraic operations with scalars. This means that vectors which are treated as entities (that is, without regard for their constituent parts) are handled in indicated operations in the same manner as scalars. For example, in direct current studies, where resistors are treated as scalars, we operate on the parallel circuit relation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

to obtain

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

and

$$R = \frac{R_1 R_2}{R_1 + R_2}. \tag{15-8}$$

In alternating current studies, where impedances are treated as vectors, we operate on the parallel circuit relation

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

to obtain

$$\frac{1}{\mathbf{Z}} = \frac{\mathbf{Z}_2 + \mathbf{Z}_1}{\mathbf{Z}_1\mathbf{Z}_2}$$

and

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}. \tag{15-9}$$

When the impedances amount simply to resistances, which resistances are of the same magnitude as those considered in the d–c case, Eq. (15–9) becomes

$$\mathbf{Z} = \frac{\mathbf{R}_1\mathbf{R}_2}{\mathbf{R}_1 + \mathbf{R}_2} = \mathbf{R}. \tag{15-10}$$

The net circuit resistance values, R and $\mathbf{R}$, obtained in Eqs. (15–8) and (15–10) are equal in magnitude. The only essential difference between Eqs. (15–8) and (15–10) is that in Eq. (15–10) the resistances are represented by directed quantities (for mathematical convenience in a-c studies), whereas in Eq. (15–8) the resistances are represented by scalars.

The similarity of indicated vector and scalar operations justifies the frequent omission in electrical engineering practice of any distinguishing marks to designate vector quantities.

Exercise 15–11

1. In an electrical circuit resistance is regarded as a real quantity and is conventionally represented by a horizontal vector, positive resistance being directed to the right and negative resistance being directed to the left. Reactance is considered as an imaginary quantity, and as a vector is directed vertically: inductive (positive) reactance upward, and capacitive (negative) reactance downward. In a series circuit the impedance is the vector sum of the resistances and the reactances of the component parts.

Find the impedance of the series circuit of Fig. 15–6 wherein the magnitudes of the individual resistances and reactances are as shown.

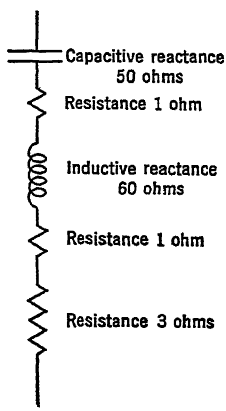


FIG. 15–6. Series circuit.

2. The diagram of Fig. 13–14(b) is designed to portray the circuit relation

$$\frac{1}{R} + \frac{1}{X} = \frac{1}{Z}. \tag{15-11}$$

Demonstrate that the construction of Fig. 13–14(b) is

correct, that is, that Eq. (15–11) is satisfied for Z as represented in the diagram.

3. The concepts of admittance, conductance, and susceptance are valuable in the solution of parallel circuit problems, since the physical relationships involved are such that for any one of these quantities, admittance, conductance, and susceptance, the total value for a parallel circuit is equal to the sum of the values for each of the branches. The admittance of a circuit in mhos (reciprocal ohms) is given by the reciprocal of the circuit impedance in ohms. The conductance is the real part of the admittance, and the susceptance is the imaginary part of the admittance.

Find the values of the admittance, the conductance, and the susceptance for each branch of the parallel circuit in Fig. 15–7 and for the complete circuit. Reactance and resistance of the circuit elements are as labelled in the diagram.

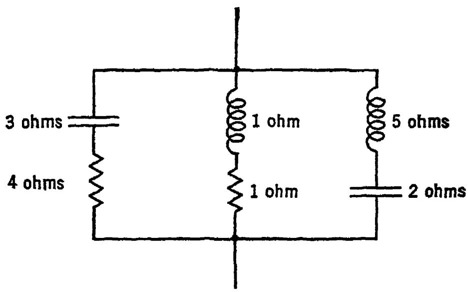


FIG. 15–7. Parallel circuit.

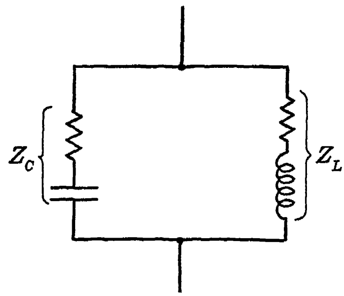


FIG. 15–8. Parallel circuit.

4. In the diagram of Fig. 15–8,

$$Z_C = R_C - jX_C,$$

and

$$Z_L = R_L + jX_L,$$

Z, the net impedance of the parallel circuit is given by the relation

$$\frac{1}{Z} = \frac{1}{Z_C} + \frac{1}{Z_L}.$$

a. Show that

$$Z = \frac{Z_C Z_L}{Z_C + Z_L}.$$

b. Show that if the resistance components of Z_C and Z_L are neglected, then at the frequency for which $X_C = X_L$ (the resonant frequency)

$$Z \approx \frac{X_L^2}{R_C + R_L} = \omega LQ,$$

where, by definition,

$$Q = \frac{X_L}{R_C + R_L}.$$

c. From the expression for Z at resonance what can be said about the nature of Z at resonance; that is, is Z resistive, capacitive, inductive . . .? How is this shown by the expression for Z?

5. A linear circuit element is one for which the voltage across the element is proportional to the current through it. Resistors, reactors, and combinations of them are linear elements. For an alternating voltage $e = E \sin \omega t$, impressed across a linear element, the current i is given by

$$i = \frac{e}{Z},$$

where Z is the impedance of the element. For e in volts and Z in ohms, i is in amperes.

Consider a current of $i = 2 \sin (100t)$ amperes in the circuit of Fig. 15-6. Find the voltage across the circuit. What is the frequency of the voltage?

Suggestion: $e = iZ$. The angle of the product vector is at each instant equal to the sum of the angles of i and of Z. i is a rotating vector; its angle varies with time. Z is a stationary vector, and its angle is constant. Thus, the angle of e at any instant differs by a constant amount from that of i; and e and i are of the same angular velocity or frequency. Since the phase difference between e and i is constant, it is sufficient in drawing the associated vectors to consider them at only one instant, for example, at time $t = 0$. The magnitude of e is given by

$$|e| = |i| \cdot |Z|.$$

6. In a parallel circuit the voltage is the same across each branch. Find the current in each of the two outside branches of the circuit of Fig. 15-7, if the current in the center branch is $5 \sin (120\pi t)$ amperes. Consider the values of resistance and reactance given in Fig. 15-7 as appropriate for a frequency of 60 cycles.

7. The instantaneous power in a circuit is given by the product of the instantaneous voltage and the instantaneous current. If a rotating vector **E** represents the instantaneous voltage through its vertical projection and a rotating vector **I** represents the instantaneous current through its vertical projection, does the product vector **EI** represent the instantaneous power through its vertical projection? Explain. (The *average* power may be obtained as indicated in Sec. 10-7.)

CHAPTER 16

LOGARITHMS

16–1. Logarithms. Because multiplying, dividing, raising to powers, and extracting of roots may often become quite laborious, numerous special schemes have been developed to simplify these computations. Perhaps the most generally useful of all such schemes is that of *logarithms*. The technique of logarithms is made possible by two facts here stated without proof, the first — as expressed in Sec. 6–6 — that the laws of exponents hold for all exponents, rational or irrational; and the second, that every positive number can be expressed as some power of a given number, where the given number, called a *base*, may be any number greater than 1.

16–2. Definition of Logarithm. A logarithm of a number is defined as the exponent of the power to which the base must be raised to equal the number. If $c = b^m$, then m is the logarithm of c to the base b. Symbolically, $\log_b c = m$. Since $3^2 = 9$, we have $\log_3 9 = 2$; and since $10^{-3} = \dfrac{1}{10^3} = 0.001$, we have $\log_{10} 0.001 = -3$. For practical numerical computations logarithms to the base 10, known as *common logarithms*, are usually employed; and the abbreviated notation $\log c$ is understood to imply $\log_{10} c$. It is to be emphasized that an appreciation of the definition of a logarithm is fundamental to an intelligent application of logarithms in computations.

Exercise 16–1

A. Evaluate:

1. $\log_3 27$.
2. $\log_4 2$.
3. $\log_{16} 16^3$.
4. $\log_2 1$.
5. $\log 100$ (base 10 understood)
6. $\log 10^5$.
7. $\log 10^{-2}$.
8. $\log 1$.

B. Show that $\log_b a = -\log_b \dfrac{1}{a}\cdot$

16–3. Theorems Concerning Logarithms. The usefulness of logarithms is a consequence of the following four theorems:

Theorem 1. The logarithm of a product is the sum of the logarithms of the factors.

Proof: Write $c = b^m$, and $d = b^n$.

Then $\log_b c = m$; $\log_b d = n$.

Now $cd = b^m \cdot b^n = b^{m+n}$,

or

$$\log_b cd = m + n;$$

that is, $\log_b cd = \log_b c + \log_b d$.

Theorem 2. The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.

Theorem 3. The logarithm of a number raised to the pth power is p times the logarithm of the number.

Theorem 4. The logarithm of the rth root of a number is $\frac{1}{r}$ times the logarithm of the number.

Exercise 16–2

Prove Theorems 2, 3, and 4 above from properties of exponents. Show specifically that Theorems 3 and 4 are valid for p and r negative, as well as for p and r positive.

16–4. Construction of a Table of Common Logarithms. In the table on the following page is listed a set of values of 10^L corresponding to various values of L. The values of 10^L corresponding to integral values of L are readily computed. Thus, for $L = -3$ we have at once

$$10^L = 10^{-3} = \frac{1}{10^3} = 0.001.$$

The particular values of 10^L shown in the table for the fractional values of L are computed as follows. Corresponding to $L = \frac{1}{2}$ we have

$$10^L = 10^{\frac{1}{2}} = \sqrt{10} = 3.162.$$

Exponent, L (logarithm)	10^L (number)
-3	0.001
-2	0.01
-1	0.1
0	1
$\frac{1}{4}$	1.778
$\frac{1}{2}$	3.162
$\frac{3}{4}$	5.623
1	10
2	100
3	1000

From this value of $\sqrt{10}$ we obtain for $L = \frac{1}{4}$

$$10^L = 10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{10}} = \sqrt{3.162} = 1.778.$$

From the above value of $10^{\frac{1}{4}}$ we obtain, in turn, for $L = \frac{3}{4}$,

$$10^L = 10^{\frac{3}{4}} = (10^{\frac{1}{4}})^3 = 1.778^3 = 5.623.$$

With sufficient patience one could construct a workable logarithm table by continuing this tabulation. The labor of such a project and the size of the resulting logarithm table could be considerably reduced by noting the following:

$$10^{\frac{5}{4}} = 10^{\frac{4}{4}} \cdot 10^{\frac{1}{4}} = 10 \cdot 10^{\frac{1}{4}}.$$

Now, as already computed in the table,

$$10^{\frac{1}{4}} = 1.778;$$

so that

$$10^{\frac{5}{4}} = 10 \cdot 1.778 = 17.78,$$

from which we have

$$\log 17.78 = \tfrac{5}{4} = \tfrac{1}{4} + 1.$$

And, likewise,

$$10^{-\frac{3}{4}} = 10^{-1} \cdot 10^{\frac{1}{4}} = \frac{1}{10} \cdot 10^{\frac{1}{4}},$$

so that

$$10^{-\frac{3}{4}} = \frac{1}{10} \cdot 1.778 = 0.1778,$$

from which we have

$$\log 0.1778 = -\tfrac{3}{4} = \tfrac{1}{4} - 1.$$

Further,

$$10^{\frac{9}{4}} = 10^2 \cdot 10^{\frac{1}{4}},$$

so that

$$10^{\frac{9}{4}} = 10^2 \cdot 1.778 = 177.8,$$

from which

$$\log 177.8 = \frac{9}{4} = \frac{1}{4} + 2.$$

These examples suggest that a table of common logarithms be restricted to logarithms of the numbers between 1 and 10, since the logarithms of all the numbers can be obtained from them. For example, from a table which lists log 1.778 as $\frac{1}{4}$ we deduce that $\log 0.0178 = \frac{1}{4} - 2$, and that $\log 1778 = \frac{1}{4} + 3$. The restricted table serves for the logarithms of numbers which are less than 1 only because of our writing the fractional part of any logarithm as positive. For example, whereas $\log 0.1778 = -\frac{3}{4}$ and $\log 0.01778 = -1\frac{3}{4}$, we choose to write $\log 0.1778 = \frac{1}{4} - 1$ and $\log 0.01778 = \frac{1}{4} - 2$. With this scheme the fraction $\frac{1}{4}$ is part of the logarithm of any number which has 1778 for its significant figures.

As a résumé of the foregoing:

$$\begin{aligned}
\log 1778 &= \tfrac{1}{4} + 3;\\
\log 177.8 &= \tfrac{1}{4} + 2;\\
\log 17.78 &= \tfrac{1}{4} + 1;\\
\log 1.778 &= \tfrac{1}{4} + 0;\\
\log 0.1778 &= \tfrac{1}{4} - 1;\\
\log 0.01778 &= \tfrac{1}{4} - 2;\\
\log 0.001778 &= \tfrac{1}{4} - 3.
\end{aligned}$$

Exercise 16–3

Plot a graph of logarithm versus number (L vs. 10^L) for numbers between 0.01 and 100.

16–5. Mantissa and Characteristic. The fractional part of a logarithm is called the *mantissa* and the integral part of a logarithm is called the *characteristic*. The characteristic of the common logarithm of any power of 10 follows at once from the definition of the common logarithm. Thus, $10^4 = 10{,}000$; $\log 10{,}000 = 4$.

The characteristic of the common logarithm of any given number other than a power of 10 may be determined with reference to the résumé of Sec. 16–4. Here we observe that the characteristic in any case may be reckoned by first expressing the given number as a power of 10 multiplied by some number between 1 and 10.

Example. 1. $1778 = 1.778 \cdot 10^3$.

The logarithm of 1778 is between 3 and 4; hence the characteristic (integral part of the logarithm to which a *positive* fraction is to be added to complete the logarithm) is 3.

Example 2. $0.01778 = 1.788 \cdot 10^{-2}$.

The logarithm of 0.01778 is between -2 and -1; hence, the characteristic (integral part of the logarithm to which a *positive* fraction is to be added to complete the logarithm) is -2.

Additional examples are given below.

Number	Number expressed as a product	Characteristic of logarithm
79	$7.9 \cdot 10^1$	1
79,000	$7.9 \cdot 10^4$	4
0.025	$2.5 \cdot 10^{-2}$	-2
0.025689	$2.5689 \cdot 10^{-2}$	-2
3.6	$3.6 \cdot 10^0$	0

Although we confine our logarithm tables to positive mantissas, it should be appreciated that logarithms with negative mantissas are perfectly legitimate and are apt to arise in computations (for example, in subtractions) involving logarithms with positive mantissas. It is essential, however for any given logarithm with a negative mantissa

that we first convert it to an equivalent logarithm with a positiv mantissa before attempting to use an ordinary logarithm table in con junction with such a logarithm. By way of illustration, in Sec. 16–: we had occasion at one point to evaluate the logarithm of 0.1778 as $-\frac{3}{4}$ In a logarithm table for the logarithm of 0.1778 we find not $-0.750($ but 0.2500, implying $0.2500 - 1$. For any computations the value: -0.7500 and $0.2500 - 1$ are equivalent. However, if we wish to finc the number corresponding to a logarithm of -0.7500, we must firs1 express this logarithm as $0.2500 - 1$ before referring to a table.

Exercise 16–4

Determine the characteristic of the logarithm of each of the following numbers:

1. 10.
2. 10,000.
3. 10^{12}.
4. 0.001.
5. 10^{-6}.
6. 30.
7. 458.3.
8. 0.0029.
9. $35.8 \cdot 10^{17}$.
10. $3.12 \cdot 10^{-8}$.

16–6. Form of Expressing Logarithms. In this work we shall adopt the convention of writing a logarithm with the characteristic preceding the mantissa if the characteristic is positive, and with the characteristic following the mantissa if the characteristic is negative. Thus, we shall write:

$$\begin{aligned}
\log 639 &= 2.8055;\\
\log 63.9 &= 1.8055;\\
\log 6.39 &= 0.8055;\\
\log 0.639 &= 0.8055 - 1;\\
\log 0.0639 &= 0.8055 - 2.
\end{aligned}$$

As an alternative scheme, in computations, we shall write:

$$\begin{aligned}
\log 0.639 &= 9.8055 - 10 \text{ instead of } 0.8055 - 1;\\
\log 0.0639 &= 8.8055 - 10 \text{ instead of } 0.8055 - 2;
\end{aligned}$$

or we shall similarly modify the form of expression of a logarithm (without modifying its value) to suit our convenience. (See Example 6 of Sec. 16–11.)

Some writers use a bar over a characteristic to denote a negative characteristic and positive mantissa. Thus,

$$\bar{2}.8055 = 0.8055 - 2.$$

	LOGARITHMS										Proportional Parts								
No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4322	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5715	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6323	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

	LOGARITHMS										Proportional Parts								
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Many students from habit of previous training prefer always to write a negative 1 characteristic as $9 - 10$. This form is widely used, particularly in published tables of logarithms of trigonometric functions. However, any student who is perturbed at the sight of a logarithm with a negative characteristic written in the apparently unorthodox style, $0.8055 - 1$, should review the fundamental ideas of logarithms until the form $0.8055 - 1$ appears just as natural as any equivalent form.

16-7. Table of Common Logarithms. Table 16-1 is a complete table of the mantissas of the common logarithms of all numbers of three significant figures, the mantissas being computed to four significant figures.

16-8. Use of the Logarithm Table for Finding Logarithms. To illustrate the use of Table 16-1 for logarithms, let us find the logarithm of 0.476. The first two significant figures, namely, 4 and 7, we locate in the left column of the table under No. Then tracing along the horizontal row corresponding to 47, we find the desired mantissa in the vertical column headed by 6. The mantissa found is 0.6776. The characteristic of the logarithm of 0.476 is -1. Thus the complete logarithm of 0.476 is $0.6776 - 1$.

Logarithms of numbers of more than three significant figures may be found from Table 16-1 by interpolation. The process is similar to that employed in using a table of trigonometric functions (Sec. 9-9). To illustrate interpolation in Table 16-1 let us find the logarithm of 31.42. From the table the mantissa of the logarithm of 31.4 is 0.4969, and the mantissa of the logarithm of 31.5 is 0.4983. The difference between these mantissas is

$$0.4983 - 0.4969 = 0.0017.$$

By interpolation, then, the mantissa of the logarithm of 31.42 is

$$0.4969 + (\tfrac{2}{10} \cdot 0.0017) = 0.4969 + 0.0003 = 0.4972;$$

and the complete logarithm of 31.42 is 1.4972.

The list of proportional parts provided in the table shows a figure 3 in the row labelled 31 and in the column headed 2 (under " proportional parts "), indicating that two tenths of 17 is 3 to one significant figure. This result can, of course, be obtained mentally without the aid of the proportional parts tabulation. However, it is of interest here to note the manner in which the proportional parts data is computed. In Sec. 16-9, when we use the logarithm table to find a number from a given

logarithm, we shall appreciate better the utility of the proportiona parts data.

For much practical communications work it is adequate simply to us the nearest figures which can be found in Table 16–1. On this basi 1.4969 may be used for log 31.42. For extended computations wherei greater accuracy is required, it is advisable to obtain a table in which th mantissas are given to more than four figures.

Exercise 16–5

With the aid of Table 16–1 find the logarithm of each of the following numbers:

1. 717.
2. 71.7.
3. 0.0717.
4. $2.90 \cdot 10^{-12}$.
5. $8.154 \cdot 10^{7}$ (Interpolate).
6. 0.3608 (Interpolate).

16–9. Antilogarithms. The number which corresponds to a given logarithm L is called the *antilogarithm* of L. The antilogarithm of 2 is 100, and the antilogarithm of $0.9315 - 7$ is $8.54 \cdot 10^{-7}$.

Interpolation is necessary to find the antilogarithm corresponding to any given logarithm whose mantissa is not included in the table. Let us find the antilogarithm of 0.4050 by interpolation.

From Table 16–1,

$$\text{antilog } 0.4065 = 2.55,$$

and

$$\text{antilog } 0.4048 = 2.54.$$

The difference between the logarithms here is

$$0.4065 - 0.4048 = 0.0017,$$

and the difference between the corresponding antilogarithms is

$$2.55 - 254 = 0.01.$$

The given logarithm, 0.4050, differing from 0.4048 by 0.0002, is seen to be $\frac{2}{17}$ of the way in the scale of numbers between 0.4048 and 0.4050. By interpolation, then,

$$\begin{aligned}\text{antilog } 0.4050 &= 2.54 + \tfrac{2}{17} \cdot 0.01 \\ &= 2.54 + 0.001 = 2.541.\end{aligned}$$

The list of proportional parts makes unnecessary the computation,

$$\tfrac{2}{7} \cdot 0.01 = 0.001,$$

since the proportional parts data give the result directly. In the same

row with the logarithm 0.4048 we find the number 2 in the proportional parts list in the column headed 1, indicating that $\frac{2}{7}$ is 0.1 to one significant figure. By thus using the proportional parts data, one usually can perform interpolation by inspection.

It will be noted that the proportional parts list contains only five columns, corresponding to proportional parts of integral tenths from one through five tenths. This is adequate provided we always take the difference between the given logarithm and the logarithm of the nearest value which is found in the table. For example, to find the antilogarithm of 0.4058 we employ the difference $0.4065 - 0.4058 = 0.0007$ (instead of the difference $0.4058 - 0.4048 = 0.0010$). Then in the same row with 0.4065 we find 7 in the proportional parts column which is headed by 4, indicating that antilog 0.4058 is equal to

$$2.55 - 0.004 = 2.546.$$

Exercise 16-6

With the aid of Table 16-1, find the antilogarithm of each of the following logarithms:

1. 0.9731.
2. 0.8459.
3. 3.8462.
4. $0.1925 - 3$.
5. $7.1925 - 10$.
6. 1.6080.

16-10. Cologarithms. The *cologarithm* of a number is the logarithm of the reciprocal of the number. The cologarithm of 100 is $\log \frac{1}{100} = -2$. The cologarithm of any number N, in general, is easiest obtained from the relation

$$\log \frac{1}{N} = \log 1 - \log N = 0 - \log N.$$

Thus,

$$\begin{aligned} \operatorname{colog} 0.1778 &= 0 - \log 0.1778 = 0 - (0.2500 - 1) \\ &= 0 - 0.2500 + 1 = 0.7500. \end{aligned}$$

In practice the cologarithm of any given number may be rapidly computed by subtracting the logarithm of the number from $10 - 10$. By way of illustration: the logarithm of 65.2 is 1.8142. The cologarithm of 65.2 is then found from the subtraction:

$$\begin{aligned} \log 1 = 0 &= 10.0000 - 10 \\ \log 65.2 &= \underline{1.8142} \\ \operatorname{colog} 65.2 &= 8.1858 - 10. \end{aligned}$$

The above subtraction may be performed mentally without formally tabulating the work by noting that the result follows on subtracting from 9 each digit of the logarithm except the last one, which is subtracted from 10. (In case the last digit or digits of the logarithm are zeros, then the last non-zero digit is subtracted from 10.)

The concept of cologarithm is useful in computations involving products and quotients of numbers. To find the logarithm of

$$\frac{28.65 \cdot 35.12 \cdot 0.9739}{1.763 \cdot 0.2135},$$

we may take either

$$(\log 28.65 + \log 35.12 + \log 0.9739) - (\log 1.763 + \log 0.2135),$$

or

$$\log 28.65 + \log 35.12 + \log 0.9739 + \operatorname{colog} 1.763 + \operatorname{colog} 0.2135.$$

The computation in the latter case, using cologarithms, is simpler than in the former case where logarithms alone are used, since the latter process involves one addition of five quantities, whereas the former process involves, first, one addition of three quantities, then one addition of two quantities, and then one subtraction.

Interpolation with cologarithms proceeds in the same manner as for logarithms. Thus, to find colog 90.83, we note that

$$\operatorname{colog} 90.8 = 9.0419 - 10$$

and

$$\operatorname{colog} 90.9 = 9.0414 - 10.$$

By interpolation we obtain

$$\operatorname{colog} 90.83 = 9.0418 - 10.$$

Exercise 16–7

With the aid of Table 16–1, find the cologarithms of the following numbers:

1. 6.00.
2. 0.212.
3. 76.2.
4. 306.4.

16–11. Computation by Logarithms. The following examples illustrate the use of logarithms for computations involving products, quotients, powers, and roots:

Example 1. Evaluate $2.33 \cdot 0.017 \cdot 4800$.

$$\begin{aligned} \log 2.33 &= 0.3674 \\ \log 0.017 &= 0.2304 - 2 \\ \log 4800 &= 3.6812 \end{aligned}$$

$$\log (2.33 \cdot 0.017 \cdot 4800) = \log 2.33 + \log 0.017 + \log 4800 = 4.2790 - 2$$

$$2.33 \cdot 0.017 \cdot 4800 = \text{antilog } 2.2790 = 190.$$

Example 2. Evaluate $2.33 \cdot 0.017 \cdot (-4800)$.
The work is performed exactly as in Example 1, regarding each factor as positive, and the appropriate sign of the product is attached to the result. Thus,

$$2.33 \cdot 0.017 \cdot (-4800) = -190.$$

Example 3. Evaluate $\dfrac{2.33 \cdot 0.017}{4800}$.

$$\begin{aligned} \log 2.33 &= 0.3674 \\ \log 0.017 &= 0.2304 - 2 \\ \text{colog } 4800 &= 6.3188 - 10 \end{aligned}$$

$$\log \frac{2.33 \cdot 0.017}{4800} = \log 2.33 + \log 0.017 + \text{colog } 4800 = 6.9166 - 12$$

$$\frac{2.33 \cdot 0.017}{4800} = \text{antilog } 0.9166 - 6 = 8.25 \cdot 10^{-6}.$$

Example 4. Evaluate 4800^7.

$$\begin{aligned} \log 4800 &= 3.6812. \\ \log 4800^7 &= 7 \log 4800 = 25.7684. \\ 4800^7 &= \text{antilog } 25.7684 = 5.87 \cdot 10^{25}. \end{aligned}$$

In order to confine our attentions to the operations with logarithms we have neglected considerations of significant figures. If in Example 4 the exponent 7 in 4800^7 is correct only to the one digit shown, then it may be adequate in evaluating $7 \log 4800$ to use 4 instead of 3.6812 for the value of $\log 4800$. In this case we have

$$\begin{aligned} 7 \log 4800 &= 7 \cdot 4 = 28. \\ 4800^7 &= \text{antilog } 28 = 10^{28}. \end{aligned}$$

In case the limits of accuracy are known, to within which the value of the exponent is correct, we might better examine the extreme possible values of

the result. Let us consider that the exponent is 7 ± 1.2.

$$\begin{array}{rlrl} 7 + 1.2 &= 8.2 & 7 - 1.2 &= 5.8 \\ 8.2000 \cdot 3.6812 &= 30.186 & 5.8000 \cdot 3.6812 &= 21.351 \\ \text{antilog } 30.186 &= 1.53 \cdot 10^{30} & \text{antilog } 21.351 &= 2.24 \cdot 10^{21} \end{array}$$

Here we can say only that the result is between $2.24 \cdot 10^{21}$ and $1.53 \cdot 10^{30}$.

In the remaining examples we shall again ignore considerations of significant figures in order to emphasize the operations with logarithms. It must be borne in mind, however, that in any practical problem the item of significant figures should be taken into account.

Example 5. Evaluate $\sqrt[3]{4800}$.

$$\begin{aligned} \log 4800 &= 3.6812. \\ \log \sqrt[3]{4800} = \tfrac{1}{3} \log 4800 &= 1.2271. \\ \sqrt[3]{4800} = \text{antilog } 1.2271 &= 16.9. \end{aligned}$$

Example 6. Evaluate $\sqrt[3]{0.480}$.

$$\begin{aligned} \log 0.480 &= 0.6812 - 1. \\ \log \sqrt[3]{0.480} = \tfrac{1}{3} \log 0.480 &= 0.2271 - \tfrac{1}{3} \\ &= 0.2271 - 1 + 0.6667 \\ &= 0.8938 - 1. \\ \sqrt[3]{0.480} = \text{antilog } 0.8938 - 1 &= 0.783. \end{aligned}$$

Alternative procedures:

$$\left[\begin{aligned} \log 0.480 &= 2.6812 - 3 \\ \tfrac{1}{3} \log 0.480 &= 0.8938 - 1. \end{aligned}\right.$$

$$\left[\begin{aligned} \log 0.480 &= 29.6812 - 30 \\ \tfrac{1}{3} \log 0.480 &= 9.8938 - 10. \end{aligned}\right.$$

In the alternative procedures the characteristic of log 0.480 is expressed in two parts, one positive and one negative, with the negative part an integral multiple of 3.

Example 7. Evaluate $3.25^{-1.8}$.

$$\begin{aligned} \log 3.25 &= 0.5119 \\ -1.8 \log 3.25 &= -0.9214 \\ &= (-1 + 1) - 0.9214 = 0.0786 - 1 \\ 3.25^{-1.8} &= \text{antilog } 0.0786 - 1 = 0.120. \end{aligned}$$

Alternative procedure:

$$3.25^{-1.8} = \frac{1}{3.25^{1.8}}$$

$$\text{colog } 3.25 = 9.4881 - 10$$

$$1.8 \text{ colog } 3.25 = 17.0786 - 18.$$

Exercise 16–8

A. Evaluate each of the following by logarithms, using Table 16–1. Do not interpolate. Ignore considerations of significant figures; that is, regard each of the numbers indicated as exact.

1. $1.4 \cdot 0.0193 \cdot 256.$
2. $\dfrac{3870}{15.6}.$
3. $\dfrac{15.6}{3870}.$
4. $297 \cdot (-95.1).$
5. $2.65^4.$
6. $0.0188^{\frac{1}{3}}.$
7. $2.84^{\frac{2}{3}}.$
8. $0.00325^{1.4}.$
9. $1.07^{-\frac{3}{2}}.$
10. $\dfrac{61.4 \cdot 27.7}{5.29^3}.$

B. Evaluate each of the following by logarithms. Consider that each of the numbers indicated represents an experimental value; and interpolate or not as is necessary to obtain an answer with as many significant figures as possible.

1. $93.61 \cdot 2.175.$
2. $\dfrac{48.00}{0.1531}.$
3. $3.024^{4.1}.$
4. $0.0048^{-0.25}.$

C. In a computation involving $\dfrac{1}{d^n}$ show that it is immaterial whether we

(a) first subtract $\log d$ from zero and then multiply this difference by n, or
(b) first multiply $\log d$ by n and then subtract this product from zero.

Exercise 16–9

1. The difference in level (in decibels) between power P_2 and P_1 is

$$n = 10 \log \frac{P_2}{P_1}.$$

Show that this difference in level is positive if, P_2 is greater than P_1, and negative if P_2 is less than P_1.

2. An amplifier has a power output of 5 watts and an input of 0.005 watt. (a) By how many decibels does the output level exceed the input? (b) If by a control arrangement the output is reduced 10 decibels, what is the new output in watts?

3. The sensation of pitch as perceived by the ear is closely related to the frequency of vibration of the mechanism which sets up the sound wave. In the scientific scale used in the design and study of communications instruments the audible vibration frequencies of the progressively higher octaves of C are taken as 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, and 16,384 reciprocal seconds. Show without reference to a logarithm table that the logarithms of these frequencies increase uniformly.

CHAPTER 17

THE SLIDE RULE

17–1. Principle of the Slide Rule. With respect to the features of multiplication and division the slide rule is essentially a logarithm table in which the logarithms are given, not in their numerical values, but as distances between markings on a scale. Addition of two logarithms — corresponding to multiplication of the associated numbers — is performed by the sliding of one logarithm scale along another to give the total distance which then represents the sum of the logarithms. For

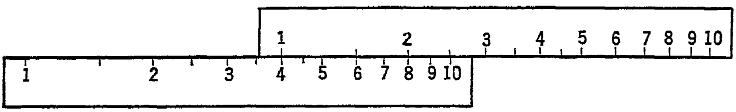


FIG. 17–1. Use of sliding logarithm scales for multiplication. (Compare Fig. 1–1.)

example, in Fig. 17–1, where the index of the upper scale is set opposite the 4 on the lower scale, the 6 on the lower scale appears under the 1.5 of the upper scale in accordance with the fact that

$$\log 4 + \log 1.5 = \log 6.$$

Further, the same setting of the slide shows that

$$\log 4 + \log 2 = \log 8,$$

and that

$$\log 4 + \log 2.5 = \log 10.$$

The slide rule graduations are labelled not as log 1, log 2, log 3 . . ., but as 1, 2, 3 . . .; and in actual usage of the slide rule we think directly in terms of products (and quotients) of numbers and not in terms of sums (and differences) of the logarithms of numbers. Thus, we interpret the setting of the slide in Fig. 17–1 as indicating

$$4 \cdot 1.5 = 6;$$
$$4 \cdot 2 = 8;$$
$$4 \cdot 2.5 = 10.$$

Or, from a different point of view, we regard the same setting as implying

$$\frac{6}{1.5} = 4;$$

$$\frac{8}{2} = 4.$$

$$\frac{10}{2.5} = 4.$$

This analysis justifies our manipulations of the slide rule for multiplications and divisions as practiced in Chapter 4 (Sec. 4-8 through 4-10).

Exercise 17-1

Perform the following computations with the aid of a slide rule, in each case using only a single setting of the slide for the entire group of products:

1. Multiply 5.24 by 9.05, 2.50, 3.96, 6.11, and 8.80.
2. Multiply 1.582 by 312, 121, 603, and 295.

17-2. Combined Operations. A combination of two operations may frequently be carried out with one setting of the slide. Thus, $\frac{63 \cdot 18}{39}$ is accomplished by setting 39 on C opposite 63 on D. Then on D, opposite 18 on C, is found the result to three significant figures, 291. The decimal point is determined by a comparison with the results of a rough approximation. Thus, $\frac{63 \cdot 18}{39} \approx \frac{60 \cdot 20}{40} = 30$. Hence, the result of $\frac{63 \cdot 18}{39} = 29.1$.

The rough calculation, in addition to determining the location of the decimal point, serves as a check on the slide rule computation. For instance, if through a mistake in manipulating the slide rule, we had obtained 58.2 as the result of $\frac{63 \cdot 18}{39}$, the approximately correct value of 30 would attest to the unreasonableness of the slide rule answer.

Exercise 17–2

Perform the following computations with the aid of a slide rule, in each case using only a single setting of the slide:

1. $\dfrac{9 \cdot 14.5}{53}$.
2. $\dfrac{5150 \cdot 101}{3.92}$.
3. $\dfrac{3.22 \cdot 0.753}{0.601}$.

17–3. Squares and Square Roots. Any of the operations of multiplication and division which can be performed on the C and D scales can likewise be performed on the A and B scales, although with reduced accuracy. The A and B scales are primarily used in conjunction with the C and D scales for squares and square roots. Any given number on the C scale has its square directly above it on the B scale, and any given number on the D scale has its square directly above it on the A scale. Square roots are obtained through the inverse process of forming squares, a number appearing on the A or B scale having its square root directly below it on the D or C scale, respectively. The square, or the square root, of a given number appears on a logarithmic scale at double, or half — as the case may be — the distance of the given number from the index (log 0) position. The upper scales are one half the size of the lower scales, so that a convenient transition to double, or half, the logarithm of a given number is accomplished through correlation of the related upper and lower scales with the indicator hairline.

Inasmuch as each of the upper scales is a double scale, there may be some question concerning the proper half of the scale to use for the square root of a particular number. This question may be settled by observing that: (a) any given number between 1 and 10 has its square root on the D scale below the given number on the left side of the A scale; and (b) any given number between 10 and 100 has its square root on the D scale below the given number on the right side of the A scale. Thus, the square root of 4, which is 2, appears on the D scale below that 4 which is on the left side of the A scale; and the square root of 40, which is 6.32, appears on the D scale below that 4 which is on the right side of the A scale. The square root of any given number, in general, may be obtained as shown in the following examples by first expressing

the number as the product of two factors: one factor, a number between 1 and 10, and the other factor, an *even* power of 10.

Example 1. Find the square root of 293.

$293 = 2.93 \cdot 10^2$. Hence, $\sqrt{293} = \sqrt{2.93} \cdot \sqrt{10^2} = 1.73 \cdot 10 = 17.3$.

Example 2.

$2930 = 29.3 \cdot 10^2$. Hence, $\sqrt{2930} = \sqrt{29.3} \cdot \sqrt{10^2} = 5.41 \cdot 10 = 54.1$.

Example 3.

$0.000293 = 2.93 \cdot 10^{-4}$. Hence, $\sqrt{0.000293} = \sqrt{2.92} \cdot \sqrt{10^{-4}} = 1.73 \cdot 10^{-2}$
$= 0.0173$.

Exercise 17-3

Perform the following computations with the aid of a slide rule, in each case manipulating only the indicator:

1. 259^2.
2. 81.4^2.
3. 0.00205^2.
4. $\sqrt{9.10}$.
5. $\sqrt{32.7 \cdot 10^{-6}}$.
6. $\sqrt{0.00910}$.
7. $\sqrt{6.62 \cdot 10^{11}}$.
8. $\sqrt{1200}$.

17-4. Square Root of the Sum of Two Squares. In alternating current studies one is often obliged to compute the square root of the sum of two squares, as $\sqrt{A^2 + B^2}$. Such a computation may be obtained with a slide rule by first expressing $\sqrt{A^2 + B^2}$ as follows:

$$\sqrt{A^2 + B^2} = A \cdot \sqrt{1 + \left(\frac{B}{A}\right)^2}. \tag{17-1}$$

The proof of Eq. (17-1) is left as an exercise (Problem A of Exercise 17-4).

To illustrate the slide rule procedure let us evaluate $\sqrt{5^2 + 7^2}$. We may express $\sqrt{5^2 + 7^2}$ either as $5 \cdot \sqrt{1 + (\frac{7}{5})^2}$ or as $7 \cdot \sqrt{1 + (\frac{5}{7})^2}$. Let us choose to use the former, that is, $5 \cdot \sqrt{1 + (\frac{7}{5})^2}$. Opposite 7 on the D scale we set 5 on the C scale to divide 7 by 5. And opposite the index of the slide we find 1.96, corresponding to $(\frac{7}{5})^2$. We then set the hairline to $1.96 + 1$, or 2.96, and we set the index of the slide under the hairline. Opposite 5 on the C scale we find the result, 8.6, on the D scale, this last operation consisting of multiplying $\sqrt{2.96}$ by 5.

Exercise 17–4

A. Prove Eq. (17–1).
B. Using the slide rule in the manner suggested in Sec. 17–4 above, evaluate:

1. $\sqrt{12.5^2 + 63.2^2}$.
2. $\sqrt{471^2 + 313^2}$.
3. $\sqrt{5.3^2 + 27^2}$.

17–5. The Inverse Scale. The *inverse scale* on a slide rule, marked CI on the rule shown in Fig. 4–5, provides for multiplication or division by the reciprocal of a number. The operation $4 \cdot \frac{1}{3}$ is performed on the rule of Fig. 4–5 by setting the index of CI to 4 on D and finding the product, 1.33, on D under 3 on CI. By the same setting, 1.33 divided by $\frac{1}{3}$ is obtained as 4.

It is this latter operation, division by the reciprocal of a number that makes possible a three-factor multiplication, such as $51 \cdot 12 \cdot 65$, with one setting of the rule; $51 \cdot 12 \cdot 65$ is treated as the equivalent problem of $(51 \div \frac{1}{12}) \cdot 65$. The value 12 on CI is set opposite 51 on D, and the result, 398, is read on D under 65 on C. The decimal point is set by inspection to give 39,800.

The quantity, $\dfrac{62}{41 \cdot 8}$, may be evaluated by setting 41 on C opposite 62 on D. The result, 189, appears on D under 8 on CI. Setting the decimal point gives

$$\frac{62}{41 \cdot 8} = 0.189.$$

On a slide rule which does not carry an inverse scale, an inverse scale may profitably be improvised for certain problems (for example, Problems B5 and B6 of Exercise 17–5) by completely removing the slide and replacing it with the same face up but with the numbers reading upside down.

Exercise 17–5

A. In terms of logarithms explain how division is accomplished on a slide rule by means of an inverse scale.
B. Perform the following computations with the aid of a slide rule, in each case using only a single setting of the slide:

1. $18 \cdot 18 \cdot 43$. (a) Compute as $(18 \div \frac{1}{18}) \cdot 43$, using the inverse scale. (b) Compute as $18^2 \cdot 43$, using the A, C, and D scales.

2. $\dfrac{52}{43 \cdot 7}$.

3. $1.53 \cdot 0.615 \cdot 83.1 \cdot 10^5$.

4. $\dfrac{1.62}{2.50 \cdot 0.316}$.

5. Divide 167 by each of the following:
   9.2, 7.4, 18.3, 36.0.

6. Divide 0.656 by each of the following:
   0.0205, 0.0319, 0.0477, 0.0590.

17–6. Folded Scales. Some slide rules are equipped with "folded" scales, labelled CF, CIF, and DF. The folded scales, CF, CIF, and DF, are identical with the regular C, CI, and D scales, respectively, except that the folded scales begin at π instead of at 1. Opposite any given number on one of the regular scales there is marked π times that number on the corresponding folded scale. The folded scales thus permit ready multiplication or division by π. Their most important advantage, however, is the elimination of frequent resettings of the slide which otherwise would be required whenever an operation involves a part of the slide that extends beyond the body of the rule. Usually an operation which is off-scale for the regular scales will be found to be on-scale for the folded scales.

Exercise 17–6

Perform the following computations with the aid of a slide rule, in each case using only a single setting of the slide. If folded scales are not available, use the A and B scales.

1. Multiply 2.80 by each of the following:
   1.50, 9.64, 4.14, 2.20.

2. Divide 75.5 by each of the following:
   11.3, 26.2, 57.9, 80.4.

17–7. Evaluation of Trigonometric Functions with a Slide Rule. An angle marked on a slide rule S scale has its sine in a corresponding position on either the A or the B scale; and an angle marked on a slide rule T scale has its tangent in a corresponding position on either the C or the D scale. Only the manner of associating angle and function differs on different rules, and with any particular rule it is usually an easy matter to verify the related angle and function scales by a few trial

settings, using angles for which the corresponding sines and tangents are known.

On some rules the S, T, B, and C scales all appear on one face of the slide, and corresponding values may be associated directly by means of the indicator hairline.

On other rules the S and T scales appear on the opposite side of the slide from the B and C scales. On a rule of this kind one scheme is to remove the slide and to replace it with the S and T scales up, as shown in Fig. 17–2. Then with indices of the slide and body aligned (as in Fig. 17–2) angles on the S scale have their tangents directly opposite on the D scale, and angles on the T scale have their tangents directly opposite on the A scale. In the rule pictured in Fig. 17–2 the setting of the hairline indicates sine 8° 10′ = 0.142 and tan 20° 40′ = 0.377.

On a rule of the " duplex " type, the indicator extends completely around the instrument and carries a hairline on the back face directly opposite the hairline on the front face. On this kind of rule an angle and its function occurring on opposite sides can be related through the paired hairlines.

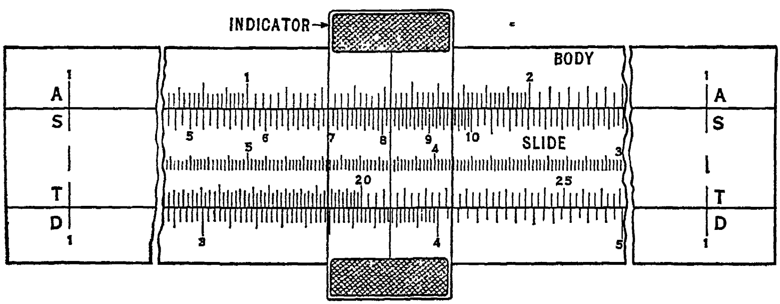


FIG. 17–2. Evaluation of trigonometric functions with one type of slide rule.

On certain other rules where the S and T scales are marked on the back of the slide, a reference scratch is provided on a little window on the back of the body of the rule. This reference scratch on the back side is directly opposite the line of the right indices on the front side, and the combination of back reference scratch and front indices line permits association of angle and function values in a manner similar to that of the paired hairlines on the " duplex " rule.

For tangents of angles, which are less than 6°, the sines are used. (See Sec. 10–2.) Tangents of angles between 45° and 90° are indicated

directly on the tangent scale on some rules. On other rules tangents of angles between 45° and 90° are obtained through the use of the relation $\tan \theta = \frac{1}{\tan (90° - \theta)}$. (Refer to Problem C1 of Exercise 9-12.) Cosines of angles are obtained in terms of sines of the complementary angles.

Exercise 17-7

A. Evaluate with the aid of a slide rule, and verify by Table 9-1:

1. sin 18.2°.
2. sin 27° 32′.
3. tan 44°.
4. tan 3°.
5. cos 85°.

B. On most slide rules it is possible to evaluate such a quantity as tan 55° with one setting of the slide. Determine how this can be done on your rule, and explain the method. Evaluate tan 55° by this method, and verify the result by reference to Table 9-1.

17-8. The Circular Slide Rule; Special Purpose Slide Rules. Not all slide rules have their basic scales identified in the same manner as that shown in Fig. 4-5, so that for a particular rule it may be necessary to make appropriate changes in the scale designations in order that the description of the manipulations given in this chapter may be applicable to that rule.

A circular slide rule is one in which the scales are marked in concentric circles. Indicators are provided in the form of radial arms which may be rotated in the manner of clock hands. The manipulations involved in any one operation are essentially the same as those employed with the straight rule.

Special slide rules are available which facilitate certain operations. The Log Log Duplex Vector and the Log Log Duplex Decitrig slide rules of Keuffel & Esser are particularly adapted to electrical circuit computations; for example, for such problems as the conversion from the rectangular form of a vector, $A + jB$, to the polar form, $M\underline{/\theta}$, and vice versa. Instruction booklets are ordinarily provided with slide rules which carry special scales so that it is unnecessary to discuss the techniques of special scales here.

CHAPTER 18

NATURAL LOGARITHMS

18–1. Exponential and Logarithmic Functions. The equations

$$y = b^x \tag{18–1}$$

and

$$x = \log_b y \tag{18–2}$$

are simply two different ways of expressing the same relation. In Eq. (18–1) y is described as an *exponential function of x to the base b*, and in Eq. (18–2) x is described as a *logarithmic function of y to the base b*. Eq. (18–1) is sometimes written:

$$y = \exp_b x.$$

Exercise 18–1

A. Write the following equations in the exponential form:

1. $x = \log_2 y$.
2. $x = \log_{3.5} y$.

B. Write the following equations in the logarithmic form:

1. $y = 3^x$.
2. $y = 2^{-x}$.
3. $y = 5^{ax}$.
4. $y = 2.7^{-3x}$.

18–2. Natural Logarithms. A particularly important type of exponential function is the exponential function to the base ϵ,

$$y = \epsilon^{kx}$$

where k is any constant and where $\epsilon = 2.718 \ldots$* The exponential

* The unusual number ϵ arises in particular in the description of processes that are characterized by a rate of change of a quantity which at each instant is proportional to the quantity. In the case of a condenser discharging through a resistor the rate at which charge leaves the plates is at each instant proportional to the magnitude of the existing charge on the plates. The result is an exponential charge decay: $q = q_0\epsilon^{-\frac{t}{k}}$, where q represents the charge on the plates at any time t, and q_0 represents the charge on the plates at time $t = 0$. k is the "time constant" of the circuit and is equal to the product of the resistance and the capacitance. (See Sec. 26–6.)

TABLE 18–1. NATURAL LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0100	0198	0296	0392	0488	0583	0677	0770	0862
1.1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740
1.2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546
1.3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293
1.4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988
1.5	0.4055	4121	4187	4253	4318	4383	4447	4511	4574	4637
1.6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247
1.7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822
1.8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366
1.9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881
2.0	0.6931	6981	7031	7080	7129	7178	7227	7275	7324	7372
2.1	7419	7467	7514	7561	7608	7655	7701	7747	7793	7839
2.2	7885	7930	7975	8020	8065	8109	8154	8198	8242	8286
2.3	8329	8372	8416	8459	8502	8544	8587	8629	8671	8713
2.4	8755	8796	8838	8879	8920	8961	9002	9042	9083	9123
2.5	0.9163	9203	9243	9282	9322	9361	9400	9439	9478	9517
2.6	9555	9594	9632	9670	9708	9746	9783	9821	9858	9895
2.7	0.9933	9969	0006	0043	0080	0116	0152	0188	0225	0260
2.8	1.0296	0332	0367	0403	0438	0473	0508	0543	0578	0613
2.9	0647	0682	0716	0750	0784	0818	0852	0886	0919	0953
3.0	1.0986	1019	1053	1086	1119	1151	1184	1217	1249	1282
3.1	1314	1346	1378	1410	1442	1474	1506	1537	1569	1600
3.2	1632	1663	1694	1725	1756	1787	1817	1848	1878	1909
3.3	1939	1969	2000	2030	2060	2090	2119	2149	2179	2208
3.4	2238	2267	2296	2326	2355	2384	2413	2442	2470	2499
3.5	1.2528	2556	2585	2613	2641	2669	2698	2726	2754	2782
3.6	2809	2837	2865	2892	2920	2947	2975	3002	3029	3056
3.7	3083	3110	3137	3164	3191	3218	3244	3271	3297	3324
3.8	3350	3376	3403	3429	3455	3481	3507	3533	3558	3584
3.9	3610	3635	3661	3686	3712	3737	3762	3788	3813	3838
4.0	1.3863	3888	3913	3938	3962	3987	4012	4036	4061	4085
4.1	4110	4134	4159	4183	4207	4231	4255	4279	4303	4327
4.2	4351	4375	4398	4422	4446	4469	4493	4516	4540	4563
4.3	4586	4609	4633	4656	4679	4702	4725	4748	4770	4793
4.4	4816	4839	4861	4884	4907	4929	4951	4974	4996	5019
4.5	1.5041	5063	5085	5107	5129	5151	5173	5195	5217	5239
4.6	5261	5282	5304	5326	5347	5369	5390	5412	5433	5454
4.7	5476	5497	5518	5539	5560	5581	5602	5623	5644	5665
4.8	5686	5707	5728	5748	5769	5790	5810	5831	5851	5872
4.9	5892	5913	5933	5953	5974	5994	6014	6034	6054	6074
5.0	1.6094	6114	6134	6154	6174	6194	6214	6233	6253	6273
5.1	6292	6312	6332	6351	6371	6390	6409	6429	6448	6467
5.2	6487	6506	6525	6544	6563	6582	6601	6620	6639	6658
5.3	6677	6696	6715	6734	6752	6771	6790	6808	6827	6845
5.4	6864	6882	6901	6919	6938	6956	6974	6993	7011	7029

TABLE 18–1. NATURAL LOGARITHMS—*Continued*

N	0	1	2	3	4	5	6	7	8	9
5.5	1.7047	7066	7084	7102	7120	7138	7156	7174	7192	7210
5.6	7228	7246	7263	7281	7299	7317	7334	7352	7370	7387
5.7	7405	7422	7440	7457	7475	7492	7509	7527	7544	7561
5.8	7579	7596	7613	7630	7647	7664	7681	7699	7716	7733
5.9	7750	7766	7783	7800	7817	7834	7851	7867	7884	7901
6.0	1.7918	7934	7951	7967	7984	8001	8017	8034	8050	8066
6.1	8083	8099	8116	8132	8148	8165	8181	8197	8213	8229
6.2	8245	8262	8278	8294	8310	8326	8342	8358	8374	8390
6.3	8405	8421	8437	8453	8469	8485	8500	8516	8532	8547
6.4	8563	8579	8594	8610	8625	8641	8656	8672	8687	8703
6.5	1.8718	8733	8749	8764	8779	8795	8810	8825	8840	8856
6.6	8871	8886	8901	8916	8931	8946	8961	8976	8991	9006
6.7	9021	9036	9051	9066	9081	9095	9110	9125	9140	9155
6.8	9169	9184	9199	9213	9228	9242	9257	9272	9286	9301
6.9	9315	9330	9344	9359	9373	9387	9402	9416	9430	9445
7.0	1.9459	9473	9488	9502	9516	9530	9544	9559	9573	9587
7.1	9601	9615	9629	9643	9657	9671	9685	9699	9713	9727
7.2	9741	9755	9769	9782	9796	9810	9824	9838	9851	9865
7.3	1.9879	9892	9906	9920	9933	9947	9961	9974	9988	*0001
7.4	2.0015	0028	0042	0055	0069	0082	0096	0109	0122	0136
7.5	2.0149	0162	0176	0189	0202	0215	0229	0242	0255	0268
7.6	0281	0295	0308	0321	0334	0347	0360	0373	0386	0399
7.7	0412	0425	0438	0451	0464	0477	0490	0503	0516	0528
7.8	0541	0554	0567	0580	0592	0605	0618	0631	0643	0656
7.9	0669	0681	0694	0707	0719	0732	0744	0757	0769	0782
8.0	2.0794	0807	0819	0832	0844	0857	0869	0882	0894	0906
8.1	0919	0931	0943	0956	0968	0980	0992	1005	1017	1029
8.2	1041	1054	1066	1078	1090	1102	1114	1126	1138	1150
8.3	1163	1175	1187	1199	1211	1223	1235	1247	1258	1270
8.4	1282	1294	1306	1318	1330	1342	1353	1365	1377	1389
8.5	2.1401	1412	1424	1436	1448	1459	1471	1483	1494	1506
8.6	1518	1529	1541	1552	1564	1576	1587	1599	1610	1622
8.7	1633	1645	1656	1668	1679	1691	1702	1713	1725	1736
8.8	1748	1759	1770	1782	1793	1804	1815	1827	1838	1849
8.9	1861	1872	1883	1894	1905	1917	1928	1939	1950	1961
9.0	2.1972	1983	1994	2006	2017	2028	2039	2050	2061	2072
9.1	2083	2094	2105	2116	2127	2138	2148	2159	2170	2181
9.2	2192	2203	2214	2225	2235	2246	2257	2268	2279	2289
9.3	2300	2311	2322	2332	2343	2354	2364	2375	2386	2396
9.4	2407	2418	2428	2439	2450	2460	2471	2481	2492	2502
9.5	2.2513	2523	2534	2544	2555	2565	2576	2586	2597	2607
9.6	2618	2628	2638	2649	2659	2670	2680	2690	2701	2711
9.7	2721	2732	2742	2752	2762	2773	2783	2793	2803	2814
9.8	2824	2834	2844	2854	2865	2875	2885	2895	2905	2915
9.9	2925	2935	2946	2956	2966	2976	2986	2996	3006	3016
10.0	2.3026									

function to the base ϵ occurs so commonly in expressions for electrical and other natural phenomena that we find it useful for purposes of computation to have available a table of logarithms to the base ϵ. Logarithms to the base ϵ are referred to as *natural logarithms* (sometimes as *hyperbolic* or as *Naperian logarithms*). We shall employ the notation ln to denote "logarithm to the base ϵ."

18–3. Table of Natural Logarithms. In Table 18–1 are given the natural logarithms of numbers between 1 and 10. In the auxiliary table, Table 18–2, are compiled the values of the natural logarithms of powers of 10.

TABLE 18–2. NATURAL LOGARITHMS OF POWERS OF 10

Number	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
Logarithm	2.3026	4.6052	6.9078	9.2103	11.5129	13.8155	16.1181	18.4207	20.7233

18–4. Computation by Natural Logarithms. The following examples will illustrate the use of Tables 18–1 and 18–2.

Example 1. Using natural logarithms, find the value of $5730 \cdot 84 \cdot 4.53$.

Table 18–1 lists logarithms of only those numbers which are between 1 and 10. For this reason we write $5730 \cdot 84 \cdot 4.53$ as $5.73 \cdot 8.4 \cdot 4.53 \cdot 10^4$. We use logarithms for the multiplication of the first three factors, and we then multiply the result by 10^4 directly. From Table 18–1:

$$\begin{aligned} \ln 5.73 &= 1.7457 \\ \ln 8.4 &= 2.1282 \\ \ln 4.53 &= 1.5107 \end{aligned}$$

Adding, we get

$$\ln 5.73 + \ln 8.4 + \ln 4.53 = 5.3846.$$

The natural procedure at this point is to find the antilogarithm of 5.3846. But the largest logarithm which appears in Table 18–1 is 2.3026.

A logarithm which is given in Table 18–1 and which may be used to obtain the desired end is $5.3846 - 4.6052 = 0.7794$. 4.6052 from Table 18–2 is the logarithm of 10^2. Hence,

$$\begin{aligned} 5.3846 - 4.602 &= \ln 5.73 + \ln 8.4 + \ln 4.53 - \ln 10^2 \\ &= \ln\left(\frac{5.73 \cdot 8.4 \cdot 4.53}{10^2}\right); \end{aligned}$$

that is, 0.7794 is the logarithm of

$$\frac{5.73 \cdot 8.4 \cdot 4.53}{10^2}.$$

From Table 18–2,

$$\text{anti ln } 0.7794 = 2.18,$$

and, thus,

$$2.18 \cdot 10^2 = 5.73 \cdot 8.4 \cdot 4.53,$$

from which we have for the product of the original factors:

$$5730 \cdot 84 \cdot 4.53 = 2.18 \cdot 10^2 \cdot 10^4 = 2.18 \cdot 10^6.$$

Example 2. Find the value of $\dfrac{0.062^4}{1410}$.

$$\frac{0.062^4}{1410} = \frac{(6.2 \cdot 10^{-2})^4}{1.41 \cdot 10^3} = \frac{6.2^4 \cdot 10^{-8}}{1.41 \cdot 10^3} = \frac{6.2^4}{1.41} \cdot 10^{-11}.$$

We first evaluate $\dfrac{6.2^4}{1.41}$:

$$\ln 6.2 = 1.8245.$$

$$\begin{aligned} 4 \ln 6.2 &= 7.2980 \\ \ln 1.41 &= \underline{0.3436} \\ 4 \ln 6.2 - \ln 1.41 &= 6.9544 \\ \ln 10^3 &= \underline{6.9078} \quad \text{(selected from Table 18–2 to provide an appropriate difference from 6.9544)} \\ 4 \ln 6.2 - \ln 1.41 - \ln 10^3 &= 0.0466 \\ \text{anti ln } 0.0466 &= 1.05. \end{aligned}$$

Thus,

$$\frac{6.2^4}{1.41} = 1.05 \cdot 10^3.$$

Then

$$\frac{0.062^4}{1410} = (1.05 \cdot 10^3) \cdot 10^{-11} = 1.05 \cdot 10^{-8}.$$

Example 3. Find the value of $\frac{1410}{0.062^4}$.

$$\frac{1510}{0.062^4} = \frac{1.41 \cdot 10^3}{(6.2 \cdot 10^{-2})^4} = \frac{1441 \cdot 10^3}{6.2^4 \cdot 10^{-8}} = \frac{1.41}{6.2^4} \cdot 10^{11}$$

$$\begin{aligned}
\ln 1.41 &= 0.3436 \\
4 \ln 6.2 &= \underline{7.2980} \\
\ln 1.41 - 4 \ln 6.2 &= -6.9544 \\
\ln 10^4 &= \underline{9.2103} \\
\ln 1.41 - 4 \ln 6.2 + \ln 10^4 &= 2.2559 \\
\text{anti}\ln 2.2559 &= 9.54 \\
\frac{1.41}{6.2} &= 9.54 \cdot 10^{-4}
\end{aligned}$$

and

$$\frac{1410}{0.062^4} = 9.54 \cdot 10^{-4} \cdot 10^{11} = 9.54 \cdot 10^7.$$

Example 4. The saturation current in amperes per square centimeter of an electron emitter is given by Richardson's equation:

$$J = AT^2 \, \epsilon^{-\frac{w}{kT}}$$

where $A = 60.2$; $k = 8.63 \cdot 10^{-5}$; T is the absolute temperature; and w is the electron affinity of the emitter. Evaluate J for an emitter of electron affinity 3.50 at a temperature of 2320°. Consider that each of the stated quantities is experimentally determined to an accuracy of three significant figures.

$$\begin{aligned}
J &= 60.2 \cdot 2320^2 \cdot \epsilon^{-\frac{3.50}{8.63 \cdot 10^{-5} \cdot 2320}} \\
&= 6.02 \cdot 10 \cdot (2.32 \cdot 10^3)^2 \cdot \epsilon^{-17.5} \\
&= 6.02 \cdot 2.32^2 \cdot 10^7 \cdot \epsilon^{-17.5}
\end{aligned}$$

$$\begin{aligned}
\ln 2.32 &= 0.85 \\
2 \ln 2.32 &= 1.6 \\
\ln 6.02 &= 1.8 \\
\ln \epsilon^{-17.5} &= \underline{-17.5} \\
2 \ln 2.32 + \ln 6.02 + \ln \epsilon^{-17.5} &= -14.0 \\
\ln 10^7 &= \underline{16.1} \\
2 \ln 2.32 + \ln 6.02 + \ln \epsilon^{-17.5} + \ln 10^7 &= 2.1 \\
6.02 \cdot 2.32^2 \, \epsilon^{-17.5} &= 7.4 \cdot 10^{-7} \\
J &= 7.4 \cdot 10^{-7} \cdot 10^7 = 7.4
\end{aligned}$$

TABLE 18-3. EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	ε^x	ε^{-x}	sinh x	cosh x	tanh x
0.00	1.000	1.0000	0.000	1.000	0.0000
0.10	1.1052	0.9048	0.100	1.005	0.0997
0.20	1.2214	0.8187	0.201	1.020	0.1974
0.30	1.3499	0.7408	0.304	1.045	0.2913
0.40	1.4918	0.6703	0.411	1.081	0.3796
0.50	1.6487	0.6065	0.521	1.128	0.4621
0.60	1.8221	0.5488	0.637	1.186	0.5371
0.70	2.0138	0.4966	0.759	1.255	0.6044
0.80	2.2255	0.4493	0.888	1.337	0.6640
0.90	2.4596	0.4066	1.026	1.433	0.7163
1.00	2.7183	0.3679	1.175	1.543	0.7616
1.10	3.0042	0.3329	1.335	1.669	0.8005
1.20	3.3201	0.3012	1.509	1.811	0.8337
1.30	3.6693	0.2725	1.698	1.971	0.8617
1.40	4.0552	0.2466	1.904	2.151	0.8854
1.50	4.4817	0.2231	2.129	2.352	0.9052
1.60	4.9530	0.2019	2.376	2.578	0.9217
1.70	5.4739	0.1827	2.646	2.828	0.9354
1.80	6.0496	0.1653	2.942	3.108	0.9468
1 90	6.6859	0.1496	3.268	3.412	0.9562
2.00	7.3891	0.1353	3.627	3.762	0.9640
2.10	8.1662	0.1225	4.022	4.144	0.9705
2.20	9.0250	0.1108	4.457	4.568	0.9757
2.30	9.9742	0.1003	4.937	5.037	0.9801
2.40	11.023	0.0907	5.466	5.557	0.9837
2.50	12.182	0.0821	6.050	6.132	0.9866
2.60	13.464	0.0743	6.695	6.770	0.9890
2.70	14.880	0.0672	7.406	7.473	0.9910
2.80	16.445	0.0608	8.192	8.253	0.9926
2.90	18.174	0.0550	9.056	9.115	0.9940

TABLE 18-3. *Continued*

x	ε^x	ε^{-x}	sinh x	cosh x	tanh x
3.00	20.086	0.0498	10.018	10.068	0.9951
3.10	22.198	0.0451	11.077	11.122	0.9960
3.20	24.533	0.0408	12.246	12.287	0.9967
3.30	27.113	0.0369	13.538	13.575	0.9973
3.40	29.964	0.0334	14.965	14.999	0.9978
3.50	33.115	0.0302	16.543	16.573	0.9982
3.60	36.598	0.0273	18.285	18.313	0.9985
3.70	40.447	0.0247	20.211	20.236	0.9988
3.80	44.701	0.0224	22.339	22.362	0.9990
3.90	49.402	0.0202	24.691	24.711	0.9992
4.00	54.598	0.0183	27.290	27.308	0.9993
4.10	60.340	0.0166	30.162	30.178	0.99945
4.20	66.686	0.0150	33.336	33.351	0.99955
4.30	73.700	0.0136	36.843	36.857	0.99963
4.40	81.451	0.0123	40.719	40.732	0.99970
4.50	90.017	0.0111	45.003	45.014	0.99975
4.60	99.484	0.0101	49.737	49.747	0.99980
4.70	109.95	0.0091	54.969	54.978	0.99983
4.80	121.51	0.0082	60.751	60.759	0.99986
4.90	134.29	0.0075	67.141	67.149	0.99989
5.00	148.41	0.0067	74.203	74.210	0.99991
5.10	164.02	0.0061	82.008	82.014	0.99993
5.20	181.27	0.0055	90.633	90.639	0.99994
5.30	200.34	0.0050	100.17	100.17	0.99995
5.40	221.41	0.0045	110.70	110.71	0.99996
5.50	244.69	0.0041	122.34	122.35	0.99997
5.60	270.43	0.0037	135.21	135.22	0.99997
5.70	298.87	0.0034	149.43	149.44	0.99998
5.80	330.30	0.0030	165.15	165.15	0.99998
5.90	365.04	0.0027	182.52	182.52	0.99998
6.00	403.43	0.0025	201.71	201.72	0.99999

Exercise 18-4

A. Plot for positive values of x:

1. ϵ^{x}. 2. ϵ^{-x}. 3. $1 - \epsilon^{-x}$.

B. From the definitions of sinh x, cosh x, and tanh x, show that:

$$\sinh 0 = 0;$$
$$\cosh 0 = 1;$$
$$\tanh 0 = 0.$$

C. Prove the following identities:

1. $\tanh x = \dfrac{\sinh x}{\cosh x}$.
2. $\sinh (-x) = -\sinh x$.
3. $\cosh (-x) = \cosh x$.
4. $\tanh (-x) = -\tanh x$.
5. $\cosh^2 x - \sinh^2 x = 1$.

18-7. Use of Table 18-3. To evaluate an exponential expression whose value is not listed directly in Table 18-3, we write the given expression as the product of two exponential functions, one of which is evaluated in Table 18-2 and the other of which is evaluated in Table 18-3. Inasmuch as the former expression is equal to a power of 10, the product of the two expressions, the desired result, is obtained at once. Examples are given below.

Example 1. Evaluate $\epsilon^{8.6}$. Consider that the value of the exponent, 8.6, is correct to two significant figures.

From Table 18-2 we find that

$$10^3 = \epsilon^{6.9}.$$

Now

$$8.6 = 6.9 + 1.7,$$

so that

$$\epsilon^{8.6} = \epsilon^{(6.9+1.7)} = \epsilon^{6.9} \cdot \epsilon^{1.7}$$
$$= 10^3 \cdot \epsilon^{1.7}.$$

It remains now only to evaluate $\epsilon^{1.7}$ from Table 18-3 and to multiply this value by 1000. $\epsilon^{1.7}$ is given in Table 18-3 is equal to 5.4739. However, we must bear in mind that the last figure in the exponent 1.7 is a doubtful figure. Let us presume that the uncertainty in this last figure, 7, is not over one unit. Then the value of the exponential according to Table 18-3 is between 4.9530 and 6.0496. At best we may write $\epsilon^{1.7} = 5.5$. We conclude, then, that

$$\epsilon^{8.6} = 5.5 \cdot 10^3.$$

Example 2. Evaluate $\epsilon^{-17.5}$. (Compare Example 4 of Sec. 18–4.) From Table 18–2,

$$10^8 = \epsilon^{18.4}.$$

Now

$$\epsilon^{-17.5} = -18.4 + 0.9,$$

so that

$$\epsilon^{-17.5} = \epsilon^{(-18.4+0.9)} = \epsilon^{-18.4} \cdot \epsilon^{0.9}$$
$$= 10^{-8} \cdot \epsilon^{0.9}.$$

From Table 18–3,

$$\epsilon^{0.9} = 2.46.$$

Hence,

$$\epsilon^{-17.5} = 2.46 \cdot 10^{-8}.$$

Exercise 18–5

With the aid of Table 18–3 evaluate the following quantities. Regard each exponent as correct to three significant figures.

1. $\epsilon^{0.505}$. 2. $\epsilon^{6.82}$. **3.** $\epsilon^{-12.6}$.

Exercise 18–6

The impedance at the sending end of a transmission line of length l is given by the relation

$$Z_s = \frac{Z_0^2 + Z_0 Z_r \coth \gamma l}{Z_r + Z_0 \coth \gamma l},$$

where Z_r is the impedance of the load connected across the receiving end, and γ and Z_0 are constants which are characteristic of the line. At radio frequencies, γ is approximately given by $\frac{j2\pi}{\lambda}$, where j is the complex operator of Chapter 15 and λ is the wavelength expressed in the same units as l.

Show that:

a. for a line which is one-half wavelength long $Z_s = Z_r$;

b. for a line which is one-quarter wavelength long $Z_s = \frac{Z_0^2}{Z_r}$.

CHAPTER 19

TRIGONOMETRIC IDENTITIES

19–1. Sine and Cosine of the Sum or Difference of Two Angles. The diagram of Fig. 19–1 permits an evaluation of the sine of the sum of two angles, or of the cosine of the sum of two angles, in terms of the

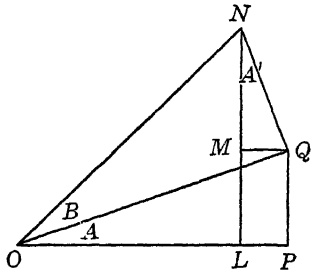


Fig. 19–1. Geometric construction for the evaluation of sin $(A + B)$.

sines and cosines of the individual angles. These relationships were stated without proof in Eqs. (9–16) and (9–17), Sec. 9–12. In Fig. 19–1, OPQ, OLN, OQN, and QMN are right angles by construction. A is angle POQ; B is angle QON; and $A + B$ is angle PON.
Further, A', that is, angle MNQ, equals A.

$$\sin (A + B) = \frac{\overline{LN}}{\overline{ON}} = \frac{\overline{LM} + \overline{MN}}{\overline{ON}} = \frac{\overline{PQ} + \overline{MN}}{\overline{ON}} = \frac{\overline{PQ}}{\overline{ON}} + \frac{\overline{MN}}{\overline{ON}}$$

$$= \frac{\overline{PQ}}{\overline{ON}} \cdot \frac{\overline{OQ}}{\overline{OQ}} + \frac{\overline{MN}}{\overline{ON}} \cdot \frac{\overline{NQ}}{\overline{NQ}}$$

$$= \frac{\overline{PQ}}{\overline{OQ}} \cdot \frac{\overline{OQ}}{\overline{ON}} + \frac{\overline{MN}}{\overline{NQ}} \cdot \frac{\overline{NQ}}{\overline{ON}};$$

or

$$\sin (A + B) = \sin A \cos B + \cos A \sin B. \tag{9–16}$$

In a similar manner,

$$\begin{aligned}\cos(A+B) &= \frac{\overline{OL}}{\overline{ON}} = \frac{\overline{OP}-\overline{LP}}{\overline{ON}} = \frac{\overline{OP}-\overline{MQ}}{\overline{ON}} = \frac{\overline{OP}}{\overline{ON}} - \frac{\overline{MQ}}{\overline{ON}} \\ &= \frac{\overline{OP}}{\overline{ON}} \cdot \frac{\overline{OQ}}{\overline{OQ}} - \frac{\overline{MQ}}{\overline{ON}} \cdot \frac{\overline{QN}}{\overline{QN}} \\ &= \frac{\overline{OP}}{\overline{OQ}} \cdot \frac{\overline{OQ}}{\overline{ON}} - \frac{\overline{MQ}}{\overline{QN}} \cdot \frac{\overline{QN}}{\overline{ON}};\end{aligned}$$

or

$$\cos(A+B) = \cos A \cos B - \sin A \sin B. \tag{9-17}$$

The above proofs are given only for acute angles (angles between 0° and 90°) and, in particular, only for those pairs of acute angles A and B which are such that the sum of A and B is less than 90°. By the construction of an appropriate diagram to supplement the diagram of Fig. 19-1, the identical steps of each of these proofs may be made to apply to cases in which A and B are each acute and $A + B$ is greater than 90°.

Further, Eqs. (9-16) and (9-17) may each be shown to be valid for angles of any magnitude. The argument involves first a consideration of two cases; (1) one angle, either A or B, acute, and the other obtuse; (2) both angles obtuse. After Eqs. (9-16) and (9-17) have both been shown to hold in each of these two cases, the generalization follows by regarding any angle as made up of the sum of two angles, one of which is a right angle. The details of case (1) above are presented herewith as an illustration of the method.

Case 1. A obtuse, B acute. Let us write $A = A_1 + 90°$, where A_1 is an acute angle. Then,

$$\begin{aligned}\sin A &= \sin(A_1 + 90°), \\ \cos A &= \cos(A_1 + 90°),\end{aligned}$$

and

$$\sin(A+B) = \sin[(A_1 + 90°) + B] = \sin[(A_1 + B) + 90°].$$

But, since the sine of the sum of any angle and 90° is equal to the cosine of the angle and, since the cosine of the sum of any angle and 90° is equal to minus the sine of the angle (see Fig. 9-5), the three above equations become

$$\begin{aligned}\sin A &= \cos A_1, \\ \cos A &= -\sin A_1,\end{aligned}$$

and

$$\sin(A + B) = \cos(A_1 + B).$$

Now, inasmuch as A_1 and B are each acute, it follows from the validity of Eq. (9-17) for acute angles that

$$\cos(A_1 + B) = \cos A_1 \cos B - \sin A_1 \sin B.$$

Thus, finally, we have

$$\begin{aligned}\sin(A + B) &= \cos(A_1 + B)\\ &= \cos A_1 \cos B - \sin A_1 \sin B\\ &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

Proofs similar to those for Eq. (9-16) and for Eq. (9-17) may be devised to show that

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \tag{19-1}$$

and

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \tag{19-2}$$

Exercise 19-1

A. Construct a diagram which is similar to that of Fig. 19-1 using acute angles A and B of such magnitudes that their sum, $A + B$, is greater than 90°. Letter the points on your diagram so that each step of the proof, which is given in Sec. 19-1 for the case of A and B both acute with $A + B < 90°$, can be construed to apply directly to your case.

B. Prove that Eq. (9-16) is valid when A and B are both obtuse.

C. Using Eqs. (9-17) and (19-2), show that

$$\cos A - \cos B = 2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right).$$

[Suggestion: write

$$P + Q = A, \text{ and } P - Q = B;$$

so that

$$P = \frac{A + B}{2}, \text{ and } Q = \frac{A - B}{2}. \quad \text{(Compare Exercise 2-2B.)}$$

Now, by Eq. (19-2),

$$\cos(P - Q) = \cos P \cos Q + \sin P \sin Q; \tag{A}$$

and, by Eq. (9–17),

$$\cos(P+Q) = \cos P \cos Q - \sin P \sin Q. \tag{B}$$

Subtract Eq. (B) above from Eq. (A).]

D. Evaluate each of the following functions, using only Eqs. (9–16), (9–17), (19–1), and (19–2) together with the relations: $\sin 30° = \cos 60° = \frac{1}{2}$, $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$, and $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$.

1. $\sin 75°$.
2. $\sin 15°$.
3. $\cos 105°$.
4. $\cos 15°$.
5. $\sin 105°$.
6. $\cos 75°$.

19–2. Condition for a Vector Angle to Be Equal to the Sum of Two Given Angles. In Sec. 15–10 we considered three vectors, $M\angle\theta$, $N\angle\phi$, and $P\angle\psi$,

where

$$MN = P,$$

and where

$$\sin\psi = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{15–2}$$

and

$$\cos\psi = \cos\theta\cos\phi - \sin\theta\sin\phi. \tag{15–3}$$

We shall here prove a statement which we made in Sec. 15–10, namely that Eqs. (15–2) and (15–3) are adequate to require that ψ equals either $\theta + \phi$ or some angle equivalent to $\theta + \phi$. The general form of the equivalent angles (all coterminal) is $\theta + \phi + (n \cdot 360°)$ where n is any integer.

Let us write S for the sum of θ and ϕ. Then we have

$$\sin S = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and

$$\cos S = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Now, regardless of what the value of S is, we see from Fig. 9–5 that the only angles which have a sine the same as that of S are of the forms $S + (n \cdot 360°)$ and $180° - S + (n \cdot 360°)$. And the only angles which have a cosine the same value as that of S are of the forms $S + (n \cdot 360°)$ and $360° - S + (n \cdot 360°)$. Of these groups the only ones which have both sine and cosine the same as that of S are those of the form

$S + (n \cdot 360°)$, in other words, those which are equivalent to S. Hence, the relations (15–2) and (15–3) require that ψ be an angle which is either $\theta + \phi$ or an equivalent (coterminal) angle.

19–3. Useful Relations. On putting $A = B = \theta$ in Eq. (9–16) and in Eq. (9–17), we obtain, respectively,

$$\sin 2\theta = 2 \sin \theta \cos \theta \tag{19–3}$$

and

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta. \tag{19–4}$$

From Eq. (9–10)

$$\cos^2 \theta = 1 - \sin^2 \theta,$$

so that Eq. (19–4) may be written as

$$\cos 2\theta = 1 - 2 \sin^2 \theta,$$

or as

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}. \tag{19–5}$$

Eqs. (9–10) and (19–4) may also be employed to yield

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}. \tag{19–6}$$

By Eq. (9–16)

$$\begin{aligned}
\sin 3\theta &= \sin (2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\
&= (2 \sin \theta \cos \theta) \cos \theta + (\cos^2 \theta - \sin^2 \theta) \sin \theta \\
&\qquad\qquad \text{[on using Eqs. (19–3) and (19–4)]} \\
&= 2 \sin \theta \cos^2 \theta + \cos^2 \theta \sin \theta - \sin^3 \theta \\
&= 3 \sin \theta \cos^2 \theta - \sin^3 \theta \\
&= 3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta \text{ (by Eq. (9–8))} \\
&= 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta \\
&= 3 \sin \theta - 4 \sin^3 \theta.
\end{aligned}$$

Or

$$\sin^3 \theta = \frac{3 \sin \theta - \sin 3\theta}{4}. \tag{19–7}$$

By a similar sequence of steps,

$$\cos^3 \theta = \frac{3 \cos \theta + \cos 3\theta}{4}. \tag{19–8}$$

On subtracting Eq. (9–17) from Eq. (19–2), we obtain

$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B,$$

or

$$\sin A \sin B = \frac{\cos (A - B) - \cos (A + B)}{2}. \tag{19–9}$$

On adding Eq. (9–17) to Eq. (19–2), we obtain

$$\cos (A - B) + \cos (A + B) = 2 \cos A \cos B,$$

or

$$\cos A \cos B = \frac{\cos (A - B) + \cos (A + B)}{2}. \tag{19–10}$$

Eqs. (19–9) and (19–10) are the same as Eqs. (9–18) and (9–19) which were originally stated without proof in Sec. 9–12.

Exercise 19–2

1. Derive Eq. (19–6) from Eqs. (9–10) and (19–4).
2. Derive Eq. (19–8) from Eqs. (9–17), (19–3), (19–4), and (9–10).
3. Prove: $\sin A \cos B = \dfrac{\sin (A - B) + \sin (A + B)}{2}$.
4. Prove: $\cos^4 \theta = \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta$.

$\left[\text{Suggestion: write } \cos^4 \theta = \cos^2 \theta \cdot \cos^2 \theta = \left(\dfrac{1 + \cos 2\theta}{2}\right) \cdot \left(\dfrac{1 + \cos 2\theta}{2}\right).\right]$

5. It can be demonstrated that any power of $\sin \theta$ or of $\cos \theta$ may be expressed as a constant term plus a series of sinusoidal terms. From average value considerations (Chapter 10), show that for any even power of $\sin \theta$ or of $\cos \theta$ the constant term in the corresponding expression cannot be zero, whereas for any odd power of $\sin \theta$ or of $\cos \theta$ the constant term must be zero.

19–4. Summary of Trigonometric Identities. Useful trigonometric identities are here collected as a group for reference.

Relations of Functions

$$\sin^2 \theta + \cos^2 \theta = 1.$$

$$1 + \tan^2 \theta = \sec^2 \theta.$$

Functions of $A \pm B$

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$
$$\sin (A - B) = \sin A \cos B - \cos A \sin B.$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B.$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 + \tan A \tan B}.$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 - \tan A \tan B}.$$

Products of Functions

$$\sin A \sin B = \tfrac{1}{2} \cos (A - B) - \tfrac{1}{2} \cos (A + B).$$
$$\cos A \sin B = \tfrac{1}{2} \cos (A - B) + \tfrac{1}{2} \cos (A + B).$$
$$\sin A \cos B = \tfrac{1}{2} \sin (A - B) + \tfrac{1}{2} \sin (A + B).$$

Double Angle Functions

$$\sin 2\theta = \sin \theta \cos \theta.$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$
$$\tan 2\theta = \frac{1 - \tan^2 \theta}{2 \tan \theta}.$$

Powers of Functions

$$\sin^2 \theta = \tfrac{1}{2} - \tfrac{1}{2} \cos 2\theta.$$
$$\cos^2 \theta = \tfrac{1}{2} + \tfrac{1}{2} \cos 2\theta.$$
$$\sin^3 \theta = \tfrac{3}{4} \cos \theta - \tfrac{1}{4} \cos 3\theta.$$
$$\cos^3 \theta = \tfrac{3}{4} \cos \theta + \tfrac{1}{4} \cos 3\theta.$$

Sums and Differences of Functions

$$\sin A + \sin B = 2 \sin \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} (A - B).$$
$$\sin A - \sin B = 2 \cos \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} (A - B).$$
$$\cos A + \cos B = 2 \cos \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} (A - B).$$
$$\cos A - \cos B = -2 \sin \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} (A - B).$$

Exercise 19–3

1. The force on the diaphragm of a telephone receiver is proportional to the square of the magnetic flux density in the air gap between the diaphragm and the electromagnet pole face. If the flux density is represented by B, the force on the diaphragm may be expressed by

$$F = kB^2, \tag{19–11}$$

where k is a constant. The core of the electromagnet is itself a permanent magnet, so that the flux density amounts to

$$B = B_1 + B_2 \sin \omega t, \tag{19–12}$$

where B_1 is the constant contribution of the permanent magnet and $B_2 \sin \omega t$ is the flux density at any time t due to a sinusoidal current of frequency $\frac{\omega}{2\pi}$ in the electromagnet. With Eq. (19–12), then Eq. (19–11) becomes

$$F = k(B_1 + B_2 \sin \omega t)^2, \tag{19–13}$$

or

$$F = k\left[B_1^2 + 2B_1B_2 \sin \omega t + \frac{B_2^2}{2}(1 - \cos 2\omega t)\right]. \tag{19–14}$$

Of the terms within the brackets in Eq. (19–14), the first corresponds to the constant (acoustically unobserved) pull on the diaphragm by the permanent magnet. The second term corresponds to a force on the diaphragm which is at the desired frequency, $\frac{\omega}{2\pi}$. The last term includes a double frequency distortion force, which is undesirable.

a. Derive Eq. (19–14) from Eq. (19–13).

b. Show that if the permanent magnet is not present, the only variational response of the receiver is at twice the desired frequency. (This means that without a permanent magnet in the receiver, the listener on a telephone circuit would hear only the octaves, and not the original tones, of every sound emitted by the speaker on the other end of the line.)

c. Show that the relative magnitude, and hence the effect, of the distortion term is lessened as the strength of the permanent magnet, that is, B_1, is increased.

2. A sinusoidally varying high-frequency voltage in the antenna of an amplitude-modulated transmitter may be represented as at (a) of Fig. 19–2 during a period when no intelligence is being conveyed, and as at (b) of Fig. 19–2 during a period of low-frequency sinusoidal modulation. The situation at

(a) may be represented at any time t by the equation

$$e = E_1 \sin \omega_1 t, \tag{19-15}$$

where e is the instantaneous voltage, E_1 is the maximum voltage, and ω_1 is 2π times the frequency. Whereas in (a) the maximum voltage, E_1, is constant, in (b) the maximum voltage, E_0, varies in accordance with the relation

$$E_0 = E_1 + E_2 \sin \omega_2 t, \tag{19-16}$$

where E_2 is the amplitude of the sinusoidal variation of the envelope of the wave, and ω_2 is 2π times the modulating signal frequency. Thus, the instantaneous voltage under modulation is given by

$$e = (E_1 + E_2 \sin \omega_2 t) \sin \omega_1 t. \tag{19-17}$$

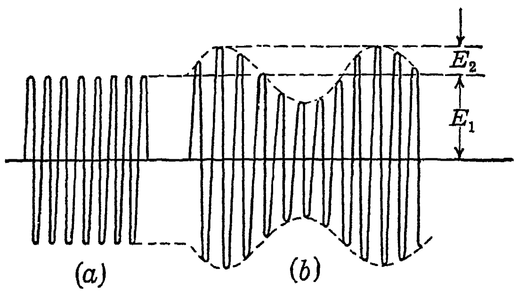


FIG. 19-2. (a) Unmodulated alternating voltage.
(b) Amplitude modulated alternating voltage.

On introduction of the modulation factor m, defined by

$$E_2 = mE_1,$$

Eq. (19-17) becomes

$$e = E_1(1 + m \sin \omega_2 t) \sin \omega_1 t, \tag{19-18}$$

which is the fundamental equation for amplitude modulation.

Trigonometric expansion of Eq. (19-18) yields the equation

$$e = E_1 \sin \omega t + \frac{mE_1}{2} \cos (\omega_1 - \omega_2)t - \frac{mE_1}{2} \cos (\omega_1 + \omega_2)t. \tag{19-19}$$

The first term in Eq. (19-19) is the same as the right side of Eq. (19-15). It represents the carrier portion of the voltage. The second and third terms in Eq. (19-19) represent voltages of frequencies $\frac{\omega_1 - \omega_2}{2\pi}$ and $\frac{\omega_1 + \omega_2}{2\pi}$, respectively, that is, voltages of frequency above and below the carrier frequency by the amount of the modulating signal frequency. This analysis indicates the

width of the frequency spectrum required for modulation, and it also shows the phase relations and relative magnitudes of the component voltages.

Derive Eq. (19–19) from Eq. (19–18). Show that, in order for modulation of the wave in Fig. 19–2 to extend to the axis, m must equal unity.

3. The non-linear current-voltage characteristics of a diode vacuum tube and of a galena crystal are advantageously employed in the demodulation of radio signals. The current i through these non-linear devices varies approximately in accordance with a relation of the type

$$i = Ae + Be^2, \tag{19–20}$$

where A and B are constants. Show that, if an amplitude-modulated voltage

$$e = E_1(1 + m \sin \omega_2 t) \sin \omega_1 t,$$

is impressed across a demodulating device which has a characteristic given by Eq. (19–20), the resultant current contains among its components a term of frequency $\dfrac{\omega_1}{2\pi}$, the desired signal frequency. (Voltage components of other frequencies are rendered ineffective by means of filtering circuits.)

4. a. Show that a beat frequency results when two sinusoidal stimuli of different frequencies are simultaneously impressed onto a device which possesses an output-input characteristic of the form

$$y = kx^2,$$

where y represents the output and x the input. (Write $x = A \sin \omega_1 t + B \sin \omega_2 t$.)

b. Show that no beat frequency results when two different frequency stimuli are simultaneously impressed onto a device which possesses a linear characteristic. (Perception by the ear of beats in musical tones is attributed to the non-linear response-stimulus characteristic of the ear.)

CHAPTER 20

LINES

20–1. Equation of a Line. Point-Slope Form. In Fig. 20–1 is shown a line which is drawn through the point (x_1, y_1). Let us presume that we know the slope of the line and designate it as m. Then, for any general point (x, y) which lies on the line,

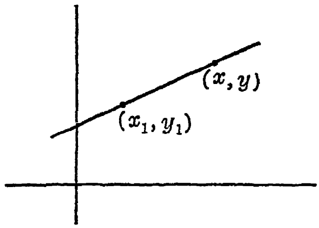


FIG. 20–1. Line through point (x_1,y_1).

$$\frac{y - y_1}{x - x_1} = m. \tag{20–1}$$

This equation, which relates the coordinates of any point on the line with the given values of x_1, y_1, and m, is then the equation of the line, since the equation is satisfied by the coordinates of every point which lies on the line and by the coordinates of no point which lies off the line. Since (x_1, y_1) is any arbitrary point and m is any arbitrary slope, Eq. (20–1) is the general equation of a line.

For the case of the particular line through the point (8,2) of slope zero, Eq. (20–1) becomes

$$\frac{y - 2}{x - 8} = 0,$$

or

$$y = 2.$$

For the case of the particular line through the origin of slope 1, Eq. (20–1) becomes

$$\frac{y - 0}{x - 0} = 1,$$

or

$$y = x.$$

The equations of these two lines, $y = 2$ and $y = x$, were obtained from a consideration of their graphs in Sec. 3–4.

Exercise 20-1

Determine the equation of each of the following lines:

1. The line through the point (2,4) with slope 3.
2. The line through the point (0,0) with slope -1.
3. The line through the point (0,3) with slope -1.
4. The line through the point $(-2,-1)$ with slope 4.

20-2. Slope-Intercept Form of the Equation of a Line. In the special case where the point at which the line crosses the y-axis has coordinates $(0,b)$, Eq. (20-1) reduces to

$$\frac{y-b}{x} = m,$$

or

$$y = mx + b. \tag{20-2}$$

Eq. (20-2) is known as the *slope-intercept* form of the equation of a line. By Eq. (20-2) a line of slope 3 and y-intercept 5 has for its equation

$$y = 3x + 5.$$

Exercise 20-2

Determine the equation of each of the following lines:

1. The line of y-intercept 2 and of slope 7.
2. The line of y-intercept -1 and of slope 5.
3. The line of y-intercept 3 and of slope -4.

20-3. Two-Point Form of the Equation of a Line. If the slope of a line is not given, but instead two points are given through which the line passes, the slope can be obtained immediately from the coordinates of the two points. Eq. (20-1) may then be extended to form the general equation of a line through two given points (x_1,y_1) and (x_2,y_2):

$$\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2}. \tag{20-3}$$

By Eq. (20-3) a line through points $(-1,7)$ and (5,3) has as its equation

$$\frac{y-7}{x+1} = \frac{7-3}{-1-5} = \frac{4}{-6} = \frac{2}{-3},$$

or

$$-3y + 21 = 2x + 2;$$

that is,

$$2x + 3y - 19 = 0.$$

Exercise 20-3

Determine the equations of the lines through the following pairs of points:

1. (4,2), (1,1).
2. (0,2), (2,0).
3. (5,3), (0,0).
4. (2,−3), (−1,2).

20-4. Vertical Lines. Eqs. (20-1), (20-2), and (20-3) are invalid to describe vertical lines. However, any vertical line is (as previously noted in Sec. 3-7) described through an equation of the type $x = k$.

Exercise 20-4

A. Why is Eq. (20-1) invalid to describe vertical lines? Eq. (20-2)? Eq. (20-3)?

B. Determine the equations of vertical lines through the following points:

1. (2,3).
2. (−5,6).
3. (0,−2).

20-5. Linear Relationship. We prove here the proposition which was first enunciated in Sec. 3-7, namely: (a) that the equation of every line is equivalent to an equation of the form $Ax + By + C = 0$, and (b) that every equation of the form $Ax + By + C = 0$ represents a line.

Let us first prove (a). The equation of a vertical line,

$$x = k,$$

is equivalent to the equation

$$x - k = 0,$$

which we recognize as of the form $Ax + By + C = 0$, wherein $A = 1$, $B = 0$, and $C = -k$.

The equation of any line other than a vertical one may be expressed through Eq. (20-2) as

$$y = mx + b,$$

or as

$$mx - y + b = 0.$$

This is of the form $Ax + By + C = 0$, wherein $A = m$, $B = -1$, and $C = b$.

To prove (b) we write $Ax + By + C = 0$, when $B \neq 0$, as

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is of the form $y = mx + b$, and therefore represents a line of slope $-\frac{A}{B}$ and of y-intercept $-\frac{C}{B}$.

When $B = 0$, we have

$$Ax + C = 0.$$

This can be expressed as

$$x = -\frac{C}{A},$$

which is immediately identified as the equation of a vertical line which is $\frac{C}{A}$ units to the left of the y-axis.

Thus, we have demonstrated (a) that any line may be represented by an equation of the form $Ax + By + C = 0$; and (b) that any equation of the form $Ax + By + C = 0$ represents a line.

Exercise 20-5

A. Show that the x-intercept of a line of equation $Ax + By + C = 0$ is $-\frac{C}{A}$.

B. By inspection of the equation indicate slope, x-intercept, and y-intercept of the line which is associated with each of the following equations:

1. $x + 2y + 3 = 0$.
2. $5x - y + 1 = 0$.
3. $-2x + 8y - 3 = 0$.
4. $4x - 7 = 0$.

20-6. Graphical Test of Linearity of an Experimental Relation. It is frequently of interest to ascertain whether or not a particular device behaves in accordance with a linear relationship. Linearity, or lack of it, in a piece of apparatus may be determined through plots of experimentally obtained values. Let us consider, for example, the following

sets of values of laboratory measurements of light flux and resulting current for a gas-filled phototube and for a vacuum phototube.

Illumination in foot-candles		0	10	20	30	40	50	60	80	110	140	175
Current in microamperes	Gas tube	0	2.7	5.5	8.0	12 0	16.1	22.0				
	Vacuum tube	0	1.0	1.9	2.9	4.2	4.8	5.6	7.5	11.0	13.2	17.0

Plotting these data in Fig. 20-2 reveals at once that the variation of current with incident flux is (1) linear over the complete range for a vacuum phototube, and (2) linear over only a limited range — up to about 25 foot-candles illumination — for the gas phototube.

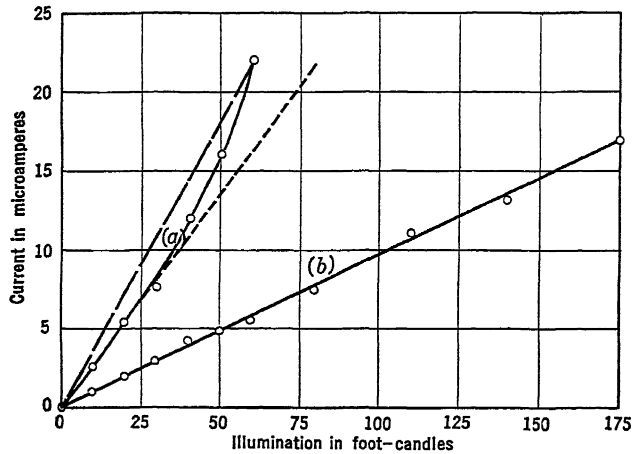


FIG. 20-2. Variation of photocurrent with illumination (a) for a gas phototube and (b) for a vacuum phototube.

In checking a set of experimental points for linearity, allowance must be made for errors in the measurements. Deviations from the true course of a curve must be expected as a result of experimental inaccuracies. The line corresponding to the vacuum tube in Fig. 20-2 has about as many points above it as below it and, thus, it is safe to conclude that the line as drawn represents the true relation between current and flux. However, for the gas tube curve it would have been unreasonable either

to have continued the straight portion as indicated by the dotted extension or to have assumed that the actual current-flux relation is as represented by the dashed line. In either of these cases the observed points are consistently to one side of the assumed true course, and the distribution of the deviations is not typical of the random nature of experimental errors.

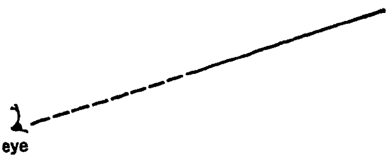


FIG. 20-3. Visual test of linearity.

20-7. Verification of Suspected Relationships. A graph is easily recognized by the eye as being either linear or non-linear because of the fact that reflected light from the graph normally travels in a linear manner to the eye. If one sights along a line (Fig. 20-3), the line appears as a point.

x	y
0	0
1.0	0.25
2.0	1.0
3.0	2.5
4.0	4.0
5.0	6 3
6.0	9.0

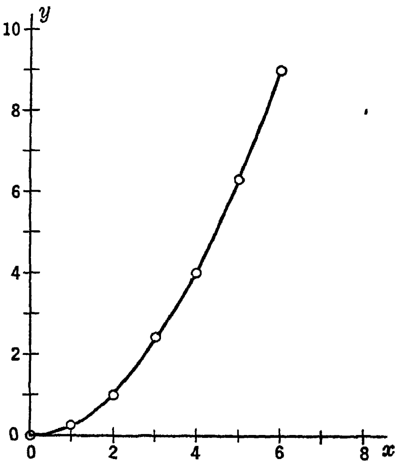


FIG. 20-4. Experimentally observed data with graph: y versus x.

We can take advantage of our ability to discern linearity or lack of it in a graph to verify suspected experimental relationships which are not linear. The experimentally observed data which are plotted in Fig. 20-4 appear to be appropriately described by an equation of the form $y = ax^2$. (Compare the graph of $y = \frac{x^2}{2}$ in Fig. 3-6). We can readily check whether or not the experimental data follow a square law ($y = ax^2$) by

TABLE 20–1. TEST FOR RELATIONSHIP BY LINEAR PLOT

Suspected relation	To test the relation, plot:
$y = \frac{a}{x} + b$	versus $\frac{1}{x}$
$\frac{1}{y} = ax + b$	$\frac{1}{y}$ versus x
$y = ax^2 + b$	y versus x^2
$y^2 = ax^2 + b$	y^2 versus x^2
$y = \frac{a}{x^2} + b$	y versus $\frac{1}{x^2}$
$\frac{1}{y} = ax^2 + b$	$\frac{1}{y}$ versus x^2
$\frac{1}{y} = \frac{a}{x^2} + b$	$\frac{1}{y}$ versus $\frac{1}{x^2}$
$\frac{1}{y^2} = ax + b$	$\frac{1}{y^2}$ versus x
$\frac{1}{y^2} = ax^2 + b$	$\frac{1}{y^2}$ versus x^2
$\frac{1}{y^2} = \frac{a}{x^2} + b$	$\frac{1}{y^2}$ versus $\frac{1}{x^2}$
$y = a\sqrt{x} + b$	y versus $\sqrt{x}$
$\sqrt{y} = ax + b$	$\sqrt{y}$ versus x
$y = \frac{a}{\sqrt{x}} + b$	y versus $\frac{1}{\sqrt{x}}$
$\frac{1}{y} = a\sqrt{x} + b$	$\frac{1}{y}$ versus $\sqrt{x}$
$\frac{1}{y} = \frac{a}{\sqrt{x}} + b$	$\frac{1}{y}$ versus $\frac{1}{\sqrt{x}}$
$\frac{1}{\sqrt{y}} = ax + b$	$\frac{1}{\sqrt{y}}$ versus x
$\sqrt{y} = a\sqrt{x} + b$	$\sqrt{y}$ versus $\sqrt{x}$
$\frac{1}{\sqrt{y}} = \frac{a}{\sqrt{x}} + b$	$\frac{1}{\sqrt{y}}$ versus $\frac{1}{\sqrt{x}}$

plotting not y versus x, but y versus x^2. We accordingly prepare a tabulation of y versus x^2 and plot these data in Fig. 20-5. Inasmuch as the

x	x^2	y
0	0	0
1.0	1.0	0.25
2.0	4.0	1.0
3.0	9.0	2.5
4.0	16.	4.0
5.0	25.	6.3
6.0	36.	9.0

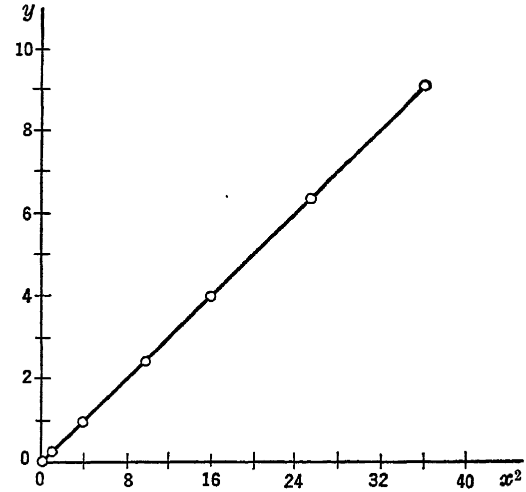


FIG. 20-5. Data of Fig. 20-4 with graph: y versus x^2.

graph of Fig. 20-5 is linear, we conclude that the given experimental data follow a relation of the form $y = ax^2$. Furthermore, from a consideration of the graph of Fig. 20-5 we can even specify the value of the constant a, which is the slope of the line. This is apparent from the manner in which a occurs in the equation $y = a(x^2)$, wherein (x^2) and y are regarded as the variables. The slope of the line of Fig. 20-5 is 0.25. Hence, the experimental data follow the relation $y = 0.25x^2$.

Various types of suspected relationships among experimental values may be verified or refuted in a manner similar to that described in the preceding paragraph. Table 20-1 suggests some functions which may be considered in this manner. All of the listed relationships are simply special cases of either $y = f(x) + b$, or of $f(y) = x + b$. To test a relation $y = f(x) + b$ we may plot y versus $f(x)$, and to test a relation $f(y) = x + b$, we may plot $f(y)$ versus x.

Exercise 20-6

1. With a constant source of potential across a variable calibrated resistor, the following current values were observed corresponding to various values

of resistance:

Resistance in ohms	100	140	230	300	350	420
Current in milliamperes	157	112	68.3	52 3	44.9	37.4

a. Plot an appropriate graph for demonstrating that Ohm's Law, $I = \frac{E}{R}$ (where E is here constant), is satisfied by these data.

b. On the same set of coordinate axes plot a graph of I versus R.

2. In a torsion balance experiment two small metallic spheres were charged with equal amounts of electricity. The following forces of repulsion were measured between the spheres at various distances of separation:

r, distance of separation between spheres in centimeters	5.3	6.8	8.1	8.5	10.5	11.6
F, force in dynes	39	24	17	15	10	8

By an appropriate plot determine whether or not the measured force follows an inverse square law.

3. In passing between two points A and B, which differ in potential by V esu, an electron of charge e esu acquires an increase of energy of Ve ergs. If the electron is initially at rest at point A, and if the only forces acting on the electron are those of the electric field, then at point B the electron has a velocity, v, in centimeters per second in the direction of the field, which is given by

$$\tfrac{1}{2}mv^2 = Ve,$$

where m is the mass of the electron in grams.

An experimental determination of velocity corresponding to various voltages yielded the following values:

Potential difference, V_{BA}, in volts	0	100	200	500	1000	2500	5000
Velocity of electron at B in billions of centimeters per second	0	5.95	8.41	13.3	18.8	29.5	41.2

Determine whether or not the experimental observations are in accordance with the theory as given. (The theoretical analysis neglects a relativity increase of mass with velocity which becomes appreciable at very high velocities.)

CHAPTER 21

POWER FUNCTIONS AND EXPONENTIAL FUNCTIONS

21–1. Power Functions. A function of the type

$$y = ax^n, \tag{21–1}$$

where a and n are constants, is known as a *power function*. An example of a power function is $y = \frac{x^2}{2}$, which was plotted in Fig. 3–6. For many applications, plotting the relation $\log y = \log (ax^n)$ is more convenient than plotting the relation $y = ax^n$. Since

$$\log (ax^n) = \log a + n \log x,$$

we have

$$\log y = \log a + n \log x. \tag{21–2}$$

For any given value of $\log x$, $\log y$ may be computed from Eq. (21–2) as equal to a constant, $\log a$, plus n times $\log x$.

Let us consider in Eq. (21–1) that $a = 8$ and $n = 1.5$. Then Eq. (21–1) becomes

$$y = 8\,x^{1.5}, \tag{21–3}$$

and Eq. (21–2) becomes

$$\begin{aligned} \log y &= \log 8 + 1.5 \log x \\ &= 0.90 + 1.5 \log x. \end{aligned} \tag{21–4}$$

Corresponding values of $\log x$ and $\log y$ for Eq. (21–4) are tabulated below. And from these tabulated values the graph of $\log y$ versus $\log x$ is plotted in Fig. 21–1.

$\log x$	-0.50	-0.30	0.00	0.30	0.60
$\log y$	0.15	0.45	0.90	1.35	1.80

This graph is a straight line, as might have been inferred from the linear nature of the equation $\log y = 0.90 + 1.5 \log x$, wherein $\log x$ is

regarded as the independent variable and log y is regarded as the dependent variable. Since the graph is a line, two points only, or the slope and a single point only, would have been sufficient for its plotting.

For any given value of log x the corresponding value of log y may be obtained directly from the appropriate point on the graph of Fig. 21-1. Or, if desired, for any given value of x the corresponding value of y may

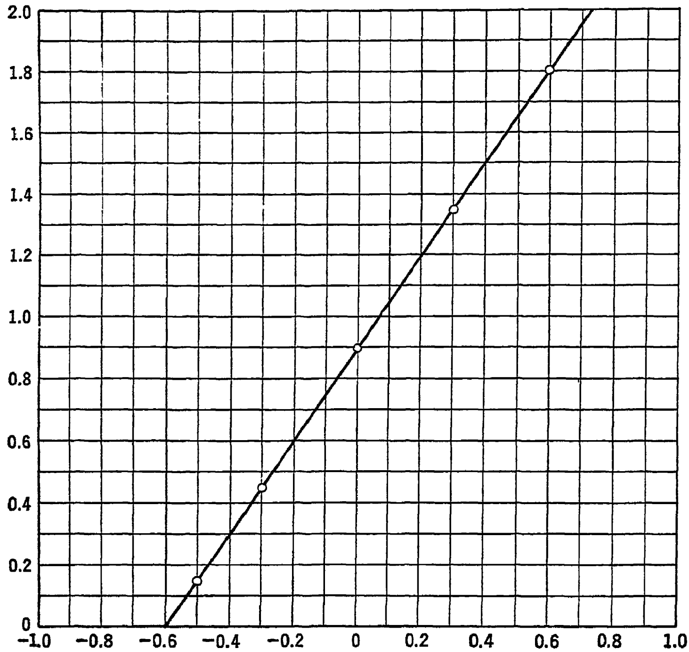


FIG. 21-1. Graph of $\log y = 0.90 + 1.5 \log x$.

be obtained by first finding log x from a logarithm table, then using the graph of Fig. 21-1 to find log y and, lastly, from log y getting the value of y as the antilogarithm. In this way the easily constructed linear graph of Fig. 21-1 serves to relate values of x and y in Eq. (21-3).

The procedure of relating values of x and y can be considerably simplified if we supplement the graph of Fig. 21-1 with two additional scales which give directly the values of x and y. The graph of Fig. 21-1 modified in this manner is reproduced in Fig. 21-2. Values denoted on the second pair of scales in Fig. 21-2 are the antilogarithms of the corresponding values on the original pair of scales.

An obvious improvement of Fig. 21-2 is (1) the deletion of the original pair of scales and the associated rulings, (2) the use of rulings which fit the second pair of scales, and (3) the elimination of any reference to logarithms. Such a graph appears in Fig. 21-3 bearing the title: "Graph of $y = 8x^{1.5}$".

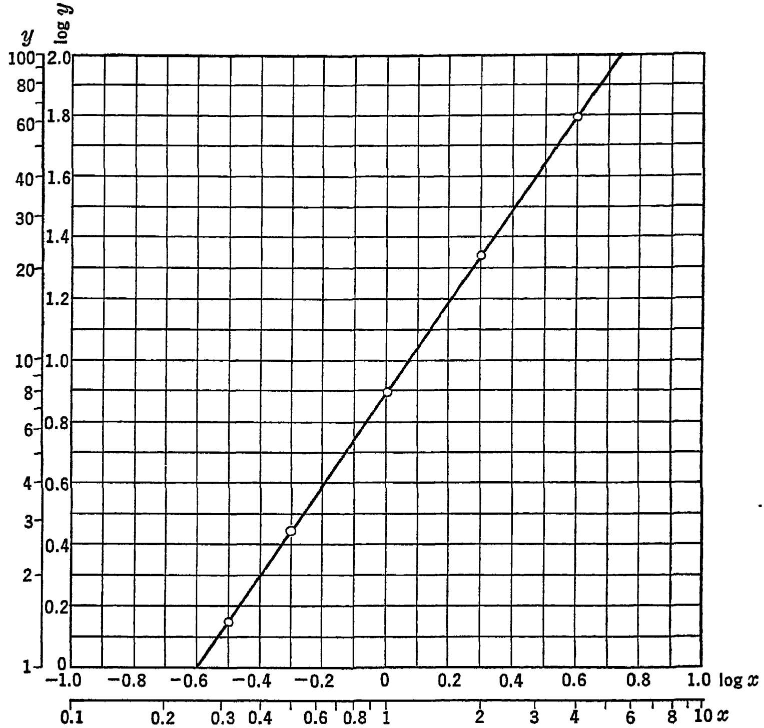


FIG. 21-2. Graph of $\log y = 0.90 + 1.5 \log x$ with additional scales of the antilogarithms.

Paper which is ruled in the manner of Fig. 21-3 is available commercially under the name of *log-log coordinate paper.* The paper used in Fig. 21-3 is described as 2-cycle by 2-cycle log-log paper. The paper used in Fig. 21-4 is called 2-cycle by 3-cycle log-log paper.

Referring to the manner of constructing the graph of the line representing $\log y = \log 8 + 1.5 \log x$ in Fig. 21-1, we see that to construct the same line on log-log paper requires (1) that we locate the point of

ordinate 8 on the vertical line through $x = 1$, and (2) that we draw a line through this point at a slope of 1.5.* The slope of 1.5 is based on a uniform scale, that is, the line rises 1.5 inches vertically for each inch of horizontal displacement. A line which is constructed in this manner may be regarded either as the graph of $\log y = \log 8 + 1.5 \log x$ or as the graph of $y = 8x^{1.5}$, depending upon whether $\log x$ and $\log y$, or x and y, are regarded as the variables.

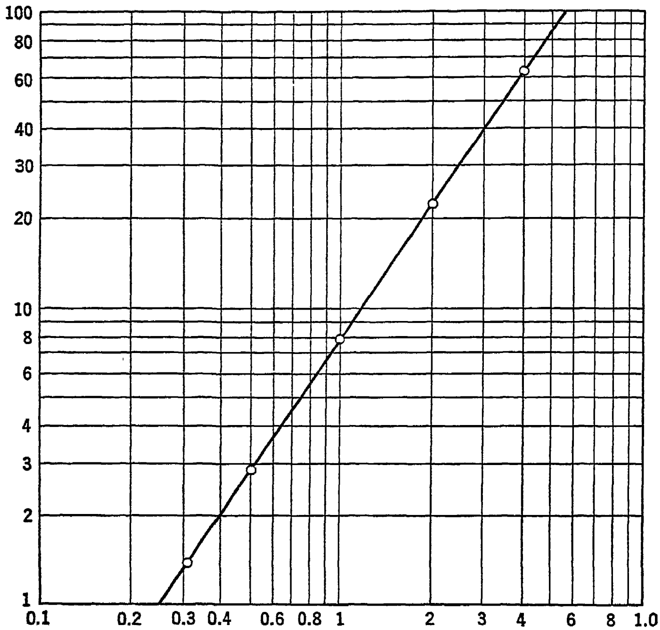


FIG. 21-3. Graph of $y = 8x^{1.5}$.

The line representing $y = \dfrac{x^{1.5}}{8}$ on log-log paper is plotted in Fig. 21-4. The slope here is the same as the slope of the line representing $y = 8x^{1.5}$ in Fig. 21-3. However, in Fig. 21-4 the intersection of the plotted line with the vertical line through $x = 1$ is at a distance of 8 units *below* the horizontal line through $y = 1$; whereas in Fig. 21-3 the intersection with the vertical line through $x = 1$ is at a distance of 8 units *above* the

* Alternative procedures are to use any other point and the slope, or any two points, in the construction of such a line.

horizontal line through $x = 1$. This is in agreement with the corresponding relations:

$$\log y = 1.5 \log x - \log 8$$

and

$$\log y = 1.5 \log x + \log 8.$$

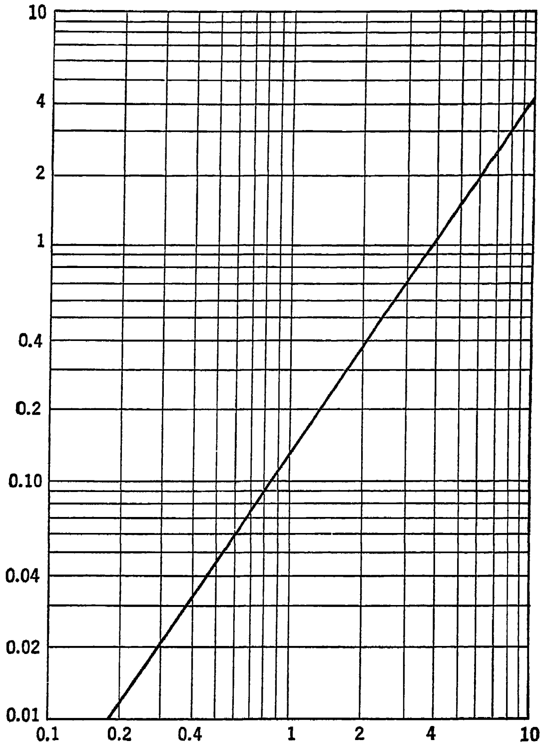


FIG. 21-4. Graph of $y = \dfrac{x^{1.5}}{8}$.

Exercise 21-1

Using log-log paper, construct graphs of the following relations:

1. $y = x^{\frac{1}{3}}$.

2. $y = 20x^3$.

3. $y = 20x^{-3}$.

4. $y = \dfrac{x^2}{5}$.

5. $y = \dfrac{x^{-2}}{5}$.

21-2. Limitations of Log-Log Paper for Power Function Graphs. Eq. (21-3) cannot be plotted on log-log paper for negative values of x because its companion relation, Eq. (21-4), which is simultaneously plotted, would then involve logarithms of negative numbers, and these are undefined. However, since the numerical quantities involved are the same for either positive or negative values of x, that is, since

$$|8x^{1.5}| = |8 \cdot (-x)^{1.5}|,$$

the graph of Fig. 21-3 may be used in either case to obtain the numerical magnitude of the result. A similar policy may be adopted as regards computations with the relation $y = -8x^{1.5}$.

Exercise 21-2

Using the graph which was constructed in Problem 3 of Exercise 21-1: (a) find the value of y corresponding to $x = -2$ in the equation $y = 20x^{-3}$; (b) find the value of y corresponding to $x = 3$ in the equation $y = -20x^{-3}$.

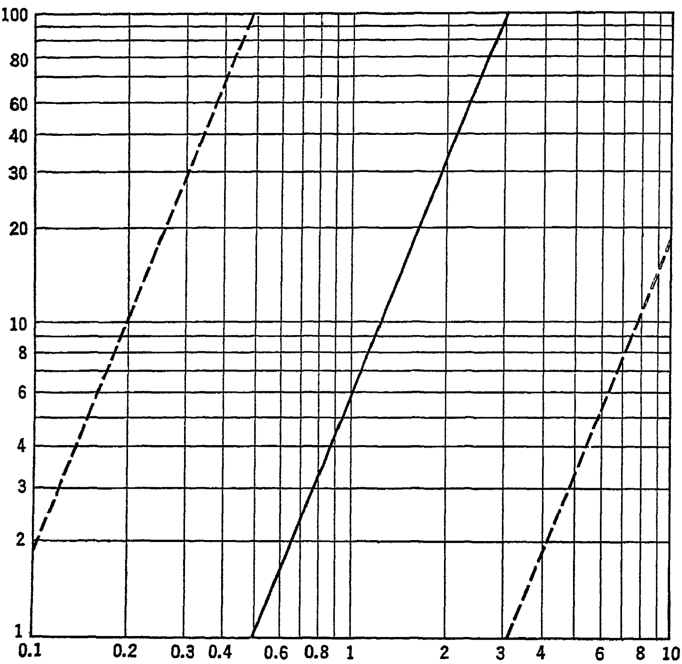


FIG. 21-5. Graph of $y = 6x^{2.5}$.

21-3. Extending the Range of a Power Function Graph on Log-Log Paper. The relation $y = 6x^{2.5}$ is graphed in Fig. 21-5. Corresponding to $x = 2$ on this graph we find $y = 34$; and corresponding to $y = 5$ we find $x = 0.93$. This graph covers only the range $y = 1$ to $y = 100$. The range might be extended by appending additional coordinate rulings at the borders of the graph, that is, by using, say, 3-cycle by 6-cycle paper. Or the range might be extended by a repetitive use of the paper, as indicated by the dotted lines.

With the latter scheme, as regards the extended (dotted) portions of the graph, the scale values supply only the correct digits, and the decimal point must be set by inspection. For example, corresponding to $x = 0.21$, we find the ordinate as given on the scale, $y = 12.1$. Setting the decimal point, we obtain, finally, $y = 0.121$. The coordinates of any point may be obtained directly without the necessity of setting the decimal point if appropriate supplementary scales are employed corresponding to each extension of the graph.

Exercise 21-3

For the following problems use 2-cycle by 2-cycle paper. If paper with more than 2-cycles by 2-cycles of rulings is all that is available, restrict yourself to only 2 cycles each way.

1. Construct the graph of $y = 6x^{-2}$ for the range $x = 1$ to $x = 100$. Using this graph: (a) find the value of y corresponding to $x = 30$; (b) find the value of x corresponding to $y = 0.15$.

2. Construct the graph of $y = x^{\frac{2}{3}}$ over the range $y = 1$ to $y = 1000$. Using this graph: (a) find the value of y corresponding to $x = 40$; (b) find the value of x corresponding to $y = 850$.

21-4. Verification of Power Function Relationship. Log-log paper provides a convenient means of obtaining a linear plot for verifying that a set of experimental values satisfies a power function relationship. Let us consider the following set of values of plate voltage, E, and of plate current, I, which were recorded for a two-element vacuum tube operating under ordinary conditions of space-charge limitation of the current.

E in volts	13	30	42	60	72	84	100
I in milliamperes	0.28	1.2	2.1	3.9	5.4	7.0	9.3

Theoretical considerations suggest that the relationship between I and E is of the form $I = kE^n$. If this theoretical relation is correct, then

points representing associated observed current and voltage values should lie on a line when they are plotted on a log-log paper. In Fig. 21-6, points representing corresponding current and voltage values are plotted on log-log paper, and these points are found to lie on a line, demonstrating that the current definitely is a power function of the voltage.

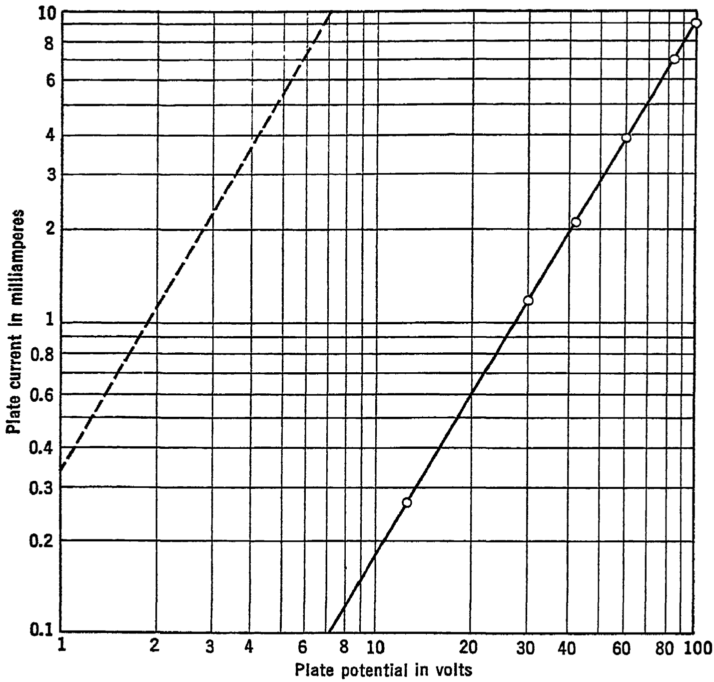


FIG. 21-6. Plot of experimental values — plate current versus plate voltage for a diode — demonstrating power function relationship.

The graph, however, does more than show simply that the current behaves according to a relation of the form, $I = kE^n$. The graph, in addition, provides the actual value of k and of n; k is the ordinate corresponding to the abscissa l, and n is the slope of the line. From Fig. 21-6, $k = 0.0034$ and $n = 1.72$. Hence, for the tube being studied, $I = 0.0034E^{1.72}$.

In the above example, k might also have been determined by using the slope together with the coordinates of any point on the line. Let us use

the point (100, 9.3). Since this point satisfies the relation $I = kE^{1.72}$ (where k is here as yet undetermined), we have

$$9.3 = k100^{1.72},$$

or

$$\log 9.3 = \log k + 1.72 \log 100;$$

that is,

$$0.97 = \log k + 1.72 \cdot 2,$$

from which

$$\log k = -2.47 = 0.53 - 3$$

and

$$k = 0.0034.$$

Furthermore, the slope, n, might have been determined analytically from the coordinates of two points on the line, just as the slope of any given line may be determined from the coordinates of two points on the line. However, here we must find first the logarithm of each of the coordinates involved. This may be apparent from a consideration of the manner in which the graph of Fig. 21-3 was constructed. Or it may be seen from the following considerations. If (E_1, I_1) and (E_2, I_2) are any two pairs of voltage and current values which satisfy the relation $I = kE^n$, then

$$I_1 = kE_1^n \tag{21-5}$$

and

$$I_2 = kE_2^n. \tag{21-6}$$

From Eq. (21-5),

$$\log I_1 = \log k + n \log E_1,$$

and from Eq. (21-6),

$$\log I_2 = \log k + n \log E_2.$$

On subtraction, we obtain

$$\begin{aligned} \log I_2 - \log I_1 &= n \log E_2 - n \log E_1 \\ &= n (\log E_2 - \log E_1), \end{aligned}$$

from which

$$n = \frac{\log I_2 - \log I_1}{\log E_2 - \log E_1}. \tag{21-7}$$

If we use the points (13, 0.28) and (100, 9.3) in Eq. (21-7) to evaluate the slope of the line of Fig. 21-6, we get

$$n = \frac{\log 9.3 - \log. 0.28}{\log 100 - \log 13}$$

$$= \frac{(0.969) - (0.447 - 1)}{2.000 - 1.114} = 1.72.$$

If the graph of Fig. 21-6 had been given instead of the set of experimental values, it would have been more convenient to select from the graph such points as (7.2, 0.10) and (27.4, 1.0). With the points (7.2, 0.10) and (27.4, 1.0), Eq. (21-7) yields.

$$n = \frac{\log 1.0 - \log 0.10}{\log 27.4 - \log 7.2}$$

$$= \frac{0 - (-1)}{1.438 - 0.857} = 1.72.$$

Exercise 21-4

Of the following sets of values of x and y, determine in which cases y is related to x by an equation of the form $y = ax^n$, and in these cases evaluate a and n.

1.

x	1.0	1.1	1.3	1.6	2.0
y	1.6	2.0	3.5	5.5	10.0

2.

x	2.0	2.5	3.0	5.0	8.0
y	58	59	59	50	25

3.

x	0.016	0.042	0.11	0.45	1.2
y	2.5	1.5	0.95	0.48	0.21

4.

x	410	600	880	1750	4200
y	18	32	57	180	600

21-5. Semi-Log Paper. Just as log-log paper is desirable for the plot of a power function, so *semi-log paper* — with one uniform scale and one

logarithmic scale — is desirable for the plot of an exponential function. Corresponding to any given equation of the form

$$y = ka^{bx}, \tag{21-8}$$

we plot (on ordinary coordinate paper of uniformly spaced rulings) the related equation

$$\log y = bx \log a + \log k, \tag{21-9}$$

wherein we regard x as the independent variable and $\log y$ as the dependent variable. The graph of Eq. (21-9) is a line, as in Fig. 21-7, of slope equal to $b \log a$ and of y-intercept equal to $\log k$. To locate the y-intercept from the given value of k it is necessary first to find $\log k$

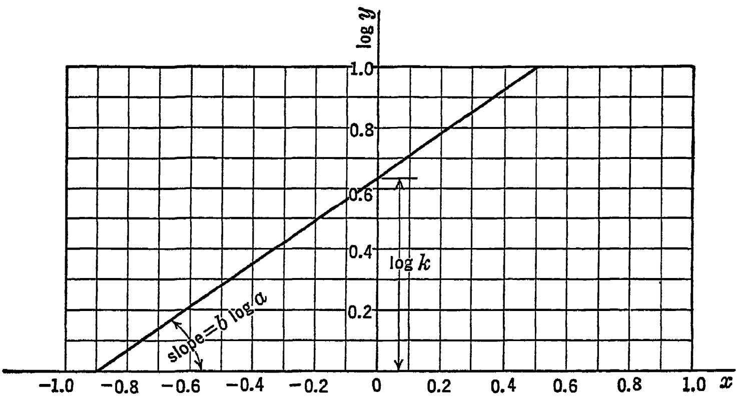


FIG. 21-7. Graph of $\log y = bx \log a + \log k$.

and then to measure this distance from the x-axis. However, if in Fig. 21-7 we had used a logarithmic vertical scale instead of a uniform vertical scale, then the locating of the y-intercept would have been simplified to selecting from the scale values the ordinate of value k. The line plotted in Fig. 21-7 may be regarded either as the graph of Eq. (21-8) or as the graph of Eq. (21-9), since it is only a matter of interpretation whether we choose to work with the variables x and y or with the variables x and $\log y$.

Using semi-log paper, the plot of the equation $y = 3\epsilon^{2x}$ is drawn in Fig. 21-8. The graph is a line of slope equal to $2 \log \epsilon = 2 \cdot 0.4343 = 0.8686$. The y-intercept is equal to $\log 3$; and $\log 3$ on the vertical logarithmic scale is at the ordinate designated by the number 3. It was

not necessary to use the y-intercept in constructing the graph of $y = 3\epsilon^{2x}$; the line might have been constructed also by using any point and the slope, or by using any two points.

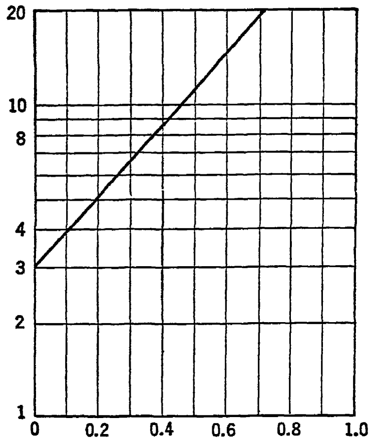


FIG. 21-8. Graph of $y = 3\epsilon^{2x}$.

Exercise 21-5

Using semi-log paper, construct graphs of the following relations.

1. $y = 10^x$.

2. $y = 10^{-x}$.

3. $y = 5 \cdot (10^{3x})$.

4. $y = 3 \cdot (10^{-0.6x})$.

5. $y = \epsilon^x$.

21-6. Verification of Exponential Relationship. We now consider the problem of checking a set of experimental data for an exponential relationship by observing whether or not the data provides a linear graph when plotted on semi-log paper. Let us examine the following set of experimental data:

x	1.2	2.8	4.4	6.2	9.2
y	2.0	2.5	3.1	4.0	6.0

These data are plotted on semi-log paper in Fig. 21-9. The linearity of the graph testifies to an exponential relationship between y and x of the form

$$y = ka^{bx}.$$

The y-intercept of the line yields for k,

$$k = 1.7.$$

And the points (1.2, 2.0) and (6.2, 4.0) give for $b \log a$ (the slope of the graph)

$$b \log a = \frac{\log 4.0 - \log 2.0}{6.2 - 1.2}$$
$$= \frac{0.602 - 0.301}{5.0} = 0.060.$$

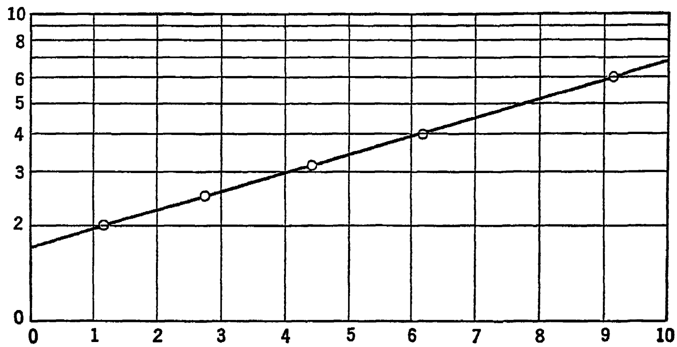


FIG. 21-9. Plot of experimental values demonstrating exponential relationship.

If we arbitrarily take $a = 10$, then $\log a = 1$, and $b = 0.060$; and the equation relating the observed quantities is

$$y = 1.7 \cdot 10^{0.060x}. \tag{21-10}$$

An equivalent equation may be formed by arbitrarily using any other number, for example, ϵ, as the base of the exponential. Since $10 = \epsilon^{2.30}$, Eq. (21-10) may be written

$$y = 1.7 \cdot (\epsilon^{2.30})^{0.060x}$$
$$= 1.7\epsilon^{2.30 \cdot 0.060x} = 1.7\epsilon^{0.14x}.$$

If the graph of Fig. 21-9 had been given instead of the set of experimental values, it would have been convenient in the determination of the slope to extrapolate the line at both ends and use the points of intersection with the horizontal lines through $y = 1.0$ and $y = 10.0$. The points are $(-3.8, 1.0)$ and $(12.8, 10.0)$; and they yield for the

slope,

$$m = b \log a = \frac{\log 10.0 - \log 1.0}{12.8 - (-3.8)}$$

$$= \frac{1.000 - 0.000}{16.6} = 0.060.$$

An alternative procedure for the determination of the slope is to use a rule to measure horizontal and vertical projections of a line segment on the graph. By such measurements we obtain $\tan \alpha = 0.30$. Then, inasmuch as 1 unit of $\log y$ occupies the same space on the graph as 5 units of x, the scale factor, f, is $\frac{1}{5}$, and the slope, m, is

$$m = f \tan \alpha = \frac{1}{5} \cdot 0.30 = 0.060.$$

Exercise 21-6

Of the following sets of values of x and y, determine in which cases y is related to x by an equation of the form $y = k\epsilon^{bx}$, and in these cases evaluate k and b.

1.

x	2.0	4.0	5.0	7.0	9.6
y	6.0	12 0	17.0	34.0	80.0

2.

x	−8.0	−5.2	0.3	3.0	5.0
y	25	36	50	120	250

3.

x	−0.50	−0.10	0.20	0.30	0.00
y	20	3.2	0.80	0.51	0.20

21-7. Use of the Logarithmic Scale for Plotting Experimental Data. On a uniform scale equal changes of a quantity occupy equal scale intervals. On a logarithmic scale equal *relative* changes occupy equal intervals. Either a change from 1 to 10 or a change from 100 to 1000 represents a tenfold increase. On a uniform scale the former increase, 1 to 10, occupies 9 units, and the latter increase, 100 to 1000, occupies 900 units. On a logarithmic scale the former increase occupies 1 unit

(log 10 − log 1 = 1 − 0 = 1), and the latter increase likewise occupies 1 unit (log 1000 − log 100 = 3 − 2 = 1). For this reason a logarithmic scale is advantageously employed wherever it is desired to emphasize the effects of relative changes in a quantity rather than the effects of actual changes.

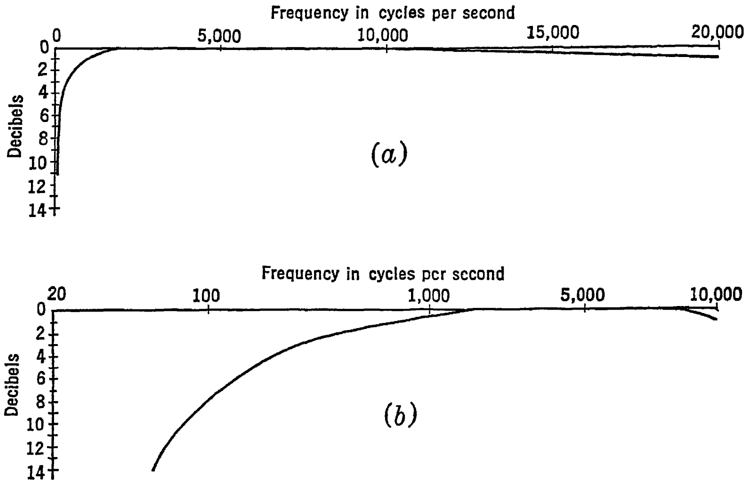


FIG. 21–10. (a) Amplifier frequency response characteristics as plotted with uniform frequency scale. (b) same characteristic as plotted with logarithmic frequency scale.

One such application of the logarithmic scale is its use for a frequency scale in plots of frequency response characteristics. The desirability of using a logarithmic frequency scale in this case is illustrated by Fig. 21–10, wherein are plotted two graphs of the same data representing the frequency response characteristic of a particular amplifier. The upper plot is made with a uniform frequency scale; the lower plot is made with a logarithmic frequency scale. The upper picture is misleading in that it unrealistically emphasizes the effective extent of the high frequencies and minimizes the effective extent of the low frequencies. In so doing it creates the impression that the response of the amplifier is good over essentially the entire audio-frequency range. The lower picture, with the logarithmic frequency scale, presents the situation in its true perspective. The logarithmic scale provides space intervals on the graph (as perceived by the eye), which are in appropriate relation to the corresponding physiological pitch intervals (as

perceived by the ear). The frequency region below 5000 cycles justly occupies about one half of the graph, and this lower picture connotes correctly that the response of the amplifier is poor over the lower half of the audible range.

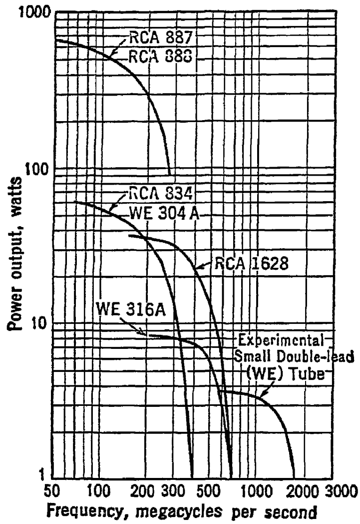


FIG. 21-11. Performance capability of ultra-high-frequency power tubes. (J. M. Stinchfield in the *Radio Engineering Handbook* edited by Henney. McGraw-Hill Book Co., 1941.)

Sometimes a logarithmic scale is used simply because it is conveniently compressed at the high end and, thus, allows the plotting of a wide range of data in a relatively small space. An example of such a use of logarithmic scales is shown in Fig. 21-11.

CHAPTER 22

DIFFERENTIATION

22–1. Objective. In Chapter 11 we defined the concept of the instantaneous rate of change of one variable, y, with respect to another variable, x, at a particular point (x,y) as that value approached by the sequence of ratios of the increments $\frac{\Delta y}{\Delta x}$, as the increment Δx approaches zero. This quantity we called the derivative, $\frac{dy}{dx}$. In mathematical symbolism we now write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \tag{22–1}$$

wherein $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is read: " the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero."

It was observed in Chapter 11 that for a given function, $y = f(x)$, the value of the derivative for any particular value of x is equal to the slope of the tangent to the graph of $y = f(x)$ at the corresponding point. The slope of the tangent varies from point to point and, thus, the value of the derivative depends upon x; in other words, the derivative is a function of x. It is our aim to formulate here a plan which will yield for any given function of x that corresponding function which represents the derivative. The process of finding the derivative of a given function is known as *differentiation*.

22–2. Increments. The graph of Fig. 22–1(a) represents a function $y = f(x)$ whose derivative at the point (x,y) we desire to compute. In order ultimately to obtain $\frac{dy}{dx}$ in terms of x through the use of Eq. (22–1), let us first express Δy in terms of x and Δx. Let us write $\Delta y = \Delta f(x)$, and for simplicity let us denote Δx by h. Then we observe from Fig. 22–1(b) that

$$\Delta f(x) = f(x + h) - f(x). \tag{22–2}$$

To illustrate Eq. (22–2) let us find $\Delta f(x)$, where $f(x) = x^2$.

$$f(x + h) = (x + h)^2 = x^2 + 2hx + h^2.$$
$$\Delta f(x) = (x + h)^2 - x^2 = x^2 + 2h + h^2 - x^2$$
$$= 2hx + h^2.$$

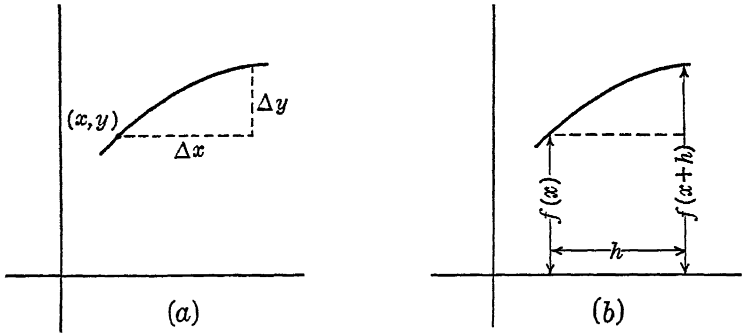


FIG. 22–1. $\Delta y = f(x + h) - f(x)$.

For a further example let us find $\Delta\left(\frac{1}{x}\right)\cdot$

Here

$$f(x) = \frac{1}{x},$$

and

$$f(x + h) = \frac{1}{x + h}\cdot$$

$$\Delta\left(\frac{1}{x}\right) = \frac{1}{x + h} - \frac{1}{x} = \frac{x}{x(x + h)} - \frac{x + h}{x(x + h)} = \frac{x - (x + h)}{x(x + h)}$$
$$= \frac{-h}{x(x + h)}\cdot$$

Exercise 22–1

Show that:

1. $\Delta(x^2 + 5) = 2hx + h^2$.
2. $\Delta(2x^3) = 6x^2h + 6xh^2 + 2h^3$.
3. $\Delta(\log_a x) = \log_a\left(1 + \frac{h}{x}\right)\cdot$

4. $\Delta(\sin x) = 2 \cos (x + \frac{1}{2}h) \sin \frac{1}{2}h$. [Write $x + h = A$ and $x = B$; and use the relation:

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B).]$$

22-3. Differentiation. We are now ready to employ Eq. (22-1) to differentiate a given function. Let us consider the function $y = x^2$ for which Eq. (22-1) becomes

$$\frac{d(x^2)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta(x^2)}{\Delta x}.$$

Writing $\Delta x = h$, we found in Sec. 22-2 that $\Delta(x^2) = 2hx + h^2$. We are interested, then, in obtaining that value which is approached by the expression $\dfrac{2hx + h^2}{h}$ as h approaches zero. Regardless of how small h is taken, it is always different from zero so that we may perform the indicated division by h to obtain

$$\frac{2hx + h^2}{h} = 2x + h.$$

The expression $2x + h$ has the limit, as h approaches zero, of $2x$. Thus,

$$\frac{d(x^2)}{dx} = 2x. \tag{22-3}$$

A graphical check on Eq. (22-3) is afforded by a plot of the derived curve which is obtained from the graph of the function $y = x^2$ in the manner of Sec. 11-6.

Let us now differentiate the function $y = \dfrac{1}{x}\cdot$ In Sec. 22-2 we found that

$$\Delta\left(\frac{1}{x}\right) = \frac{-h}{x(x + h)}.$$

Then, by Eq. (22-1),

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h \to 0} \frac{\dfrac{-h}{x(x + h)}}{h} = \lim_{h \to 0} \frac{-1}{x(x + h)}$$

$$= -\frac{1}{x^2}. \tag{22-4}$$

Exercise 22-2

A. Plot the curves of $y = x^2$ and $y = \frac{1}{x}$ over the interval $x = \frac{1}{5}$ to $x = 5$. By measuring the slopes of tangents to these curves, verify the values obtained from Eqs. (22-3) and (22-4) for the derivatives of x^2 and $\frac{1}{x}$, respectively, for each of the following values of x: $\frac{1}{4}$, 1, 4.

B. Employing the method illustrated in Sec. 22-3, show that:

1. $\frac{d}{dx} x = 1.$
2. $\frac{d}{dx}(kx) = k$, where k is a constant.
3. $\frac{d}{dx} k = 0.$

C. Differentiate:

1. $3x - 1$. 2. $2x^2 + 5x$. 3. $\frac{2}{x}$.

22-4. Summary of Results. The method of differentiation illustrated in Sec. 22-3, together with the theorems of Sec. 22-5, may be employed to differentiate various functions. Some of the more important results are stated below for reference.

$$\frac{d}{dx} x = 1. \tag{22-5}$$

$$\frac{d}{dx}(kx) = k. \tag{22-6}$$

$$\frac{d}{dx} k = 0. \tag{22-7}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}. \tag{22-8}$$

$$\frac{d}{dx}(ku) = k\frac{du}{dx}. \tag{22-9}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \tag{22-10}$$

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}. \tag{22-11}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{22-12}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v_2}. \tag{22-13}$$

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}. \tag{22-14}$$

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}. \tag{22-15}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}. \tag{22-16}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}. \tag{22-17}$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}. \tag{22-18}$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u\frac{du}{dx}. \tag{22-19}$$

$$\frac{d}{dx}(\epsilon^x) = \epsilon^x. \tag{22-20}$$

$$\frac{d}{dx}(\epsilon^u) = \epsilon^u\frac{du}{dx}. \tag{22-21}$$

$$\frac{d}{dx}(a^u) = a^u \ln a\frac{du}{dx}. \tag{22-22}$$

$$\frac{d}{dx}(\log_a u) = \frac{\log_a \epsilon}{u}\frac{du}{dx}. \tag{22-23}$$

In the above equations k, n, and a are constants; u and v are functions of x; and $\epsilon = 2.718\ldots$ All angles are measured in radians.

Applications of these equations are given in the following examples.

Example 1. Differentiate $x^{1.3}$.
By Eq. (22-8).

$$\frac{d}{dx}(x^{1.3}) = 1.3x^{1.3-1}$$
$$= 1.3x^{0.3}.$$

Example 2. Differentiate $\frac{-5}{x^4}\cdot$

Write

$$\frac{1}{x^4} = u.$$

Then

$$\frac{-5}{x^4} = -5u,$$

and, by Eq. (22-9),

$$\frac{d}{dx}(-5u) = -5\frac{du}{dx}\cdot$$

Now, by Eq. (22-8),

$$\frac{du}{dx} = \frac{d}{dx}(x^{-4}) = -4x^{-3} = -\frac{4}{x^3}\cdot$$

Thus

$$\frac{d}{dx} = \left(\frac{-5}{x^4}\right) = -5\cdot\left(\frac{-4}{x^3}\right)$$
$$= \frac{20}{x^3}\cdot$$

Example 3. Differentiate $x^3 - 6x$.
Write

$$x^3 = u$$

and

$$-6x = v.$$

Then, by Eq. (22-11),

$$\frac{d}{dx}(x^3 - 6x) = \frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$
$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(-6x).$$

By Eq. (22-8),

$$\frac{d}{dx}(x^3) = 3x^2,$$

and, by Eq. (22–6),

$$\frac{d}{dx}(-6x) = -6.$$

Thus

$$\frac{d}{dx}(x^3 - 6x) = 3x^2 - 6.$$

Example 4. Differentiate $\sin(2x + \pi)$.
Write

$$2x + \pi = u.$$

By Eq. (22–14),

$$\frac{d}{dx}\sin(2x + \pi) = \cos u \frac{du}{dx}\cdot$$

Now

$$\frac{du}{dx} = \frac{d}{dx}(2x + \pi),$$

and, by Eq. (22–11),

$$\frac{d}{dx}(2x + \pi) = \frac{d}{dx}(2x) + \frac{d\pi}{dx}\cdot$$

By Eq. (22–6),

$$\frac{d}{dx}(2x) = 2,$$

and, by Eq. (22–7),

$$\frac{d\pi}{dx} = 0.$$

Hence,

$$\frac{d}{dx}(\sin 2x + \pi) = 2\cos(2x + \pi).$$

Example 5. Differentiate $(x + \tan x)^3$.
Write

$$x + \tan x = u.$$

Then

$$(x + \tan x)^3 = u^3.$$

By Eq. (22–10),

$$\frac{d}{dx}(x + \tan x)^3 = \frac{d}{dx}(u^3) = 3u^2 \frac{du}{dx}\cdot$$

By Eq. (22–11),

$$\frac{du}{dx} = \frac{d}{dx}(x + \tan x) = \frac{dx}{dx} + \frac{d}{dx}(\tan x).$$

By Eq. (22–16),

$$\frac{d}{dx}(\tan x) = \sec^2 x \frac{dx}{dx},$$

and, by Eq. (22–5),

$$\frac{dx}{dx} = 1.$$

Hence,

$$\frac{du}{dx} = \frac{d}{dx}(x + \tan x) = 1 + \sec^2 x,$$

and

$$\frac{d}{dx}(x + \tan x)^3 = 3(x + \tan x)^2(1 + \sec^2 x).$$

Exercise 22–3

Obtain the following derivatives by using the relations of Sec. 22–4:

1. $\dfrac{d}{dx}(x^4)$.
2. $\dfrac{d}{dx}\left(\sqrt{x}\right)$.
3. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right)$.
4. $\dfrac{d}{dz}(5z^2)$.
5. $\dfrac{d}{dr}(3r^2 + 2r^4)$.
6. $\dfrac{d}{ds}(7s^5 - s^{2/3} + 3)$.
7. $\dfrac{d}{dx}\left(\dfrac{1}{1 + x^2}\right)$.
8. $\dfrac{d}{dx}(\sin^2 x)$.
9. $\dfrac{d}{d\phi}(\sin 2\phi)$.
10. $\dfrac{d}{dx}(\sin mx)$.
11. $\dfrac{d}{dx}(\cos mx)$.
12. $\dfrac{d}{d\theta}(\theta + \tan \theta)$.
13. $\dfrac{d}{d\theta}(\sin \theta \cos \theta)$.
14. $\dfrac{d}{d\theta}\left(\dfrac{\sin 2\theta}{\cos \theta}\right)$.
15. $\dfrac{d}{dt}(\epsilon^{kt})$.
16. $\dfrac{d}{dx}(\epsilon^{Ax+B})$.
17. $\dfrac{d}{dt}(5t^2 + \epsilon^{-3t})$.
18. $\dfrac{d}{dx}[\ln (ax - 1)]$.

22-5. Useful Theorems on Derivatives. The following useful theorems on derivatives are here stated without proof.

Theorem 1. If y is a function of z, where z is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}. \tag{22-24}$$

Theorem 2.

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}. \tag{22-25}$$

These theorems are applied in the examples below.

Example 1. Find, $\dfrac{dy}{dx}$, where $y = \dfrac{1}{z}$ and $z = 1 - 3x$.

$$\frac{dy}{dz} = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2},$$

and

$$\frac{dz}{dx} = \frac{d}{dx}(1 - 3x) = -3.$$

Then, by Eq. (22-24),

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{3}{z^2}$$

$$= \frac{3}{(1 - 3x)^2}.$$

Example 2. Find $\dfrac{dx}{dy}$, where $y = \sqrt{x}$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

Then, by Eq. (22-25),

$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}} = 2\sqrt{x}.$$

The result may be checked by first expressing x as a function of y and then differentiating directly. From

$$y = \sqrt{x},$$

it follows that

$$x = y^2.$$

Differentiating, we obtain

$$\frac{dx}{dy} = 2y.$$

But

$$y = \sqrt{x}.$$

Hence,

$$\frac{dx}{dy} = 2\sqrt{x}.$$

The pair of functions, $y = \sqrt{x}$ and $x = y^2$, are called *inverse functions.* Other pairs of inverse functions are $y = \sin^{-1} x$ and $x = \sin y$, $y = \log_a x$ and $x = a^y$.

Example 3. Find $\dfrac{d}{dx}(\sin^{-1} x)$.

Writing

$$y = \sin^{-1} x,$$

we have

$$x = \sin y.$$

Then

$$\frac{dx}{dy} = \cos y,$$

and, by Eq. (22–25),

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\cos\,(\sin^{-1} x)}$$

$$= \frac{1}{\sqrt{1 - x^2}}.$$

Exercise 22–4

A. Using Eq. (22–24), differentiate:

1. $3(x + 2)^2 + \dfrac{1}{\sqrt{x + 2}}$. (Write $x + 2 = z$.)
2. $\dfrac{a}{2}\left(\epsilon^{\frac{x}{a}} - \epsilon^{-\frac{x}{a}}\right)$. $\left(\text{Write } \dfrac{x}{a} = z.\right)$

B. Using Eq. (22–25), find $\frac{dy}{dx}$; given:

1. $x = \ln y$.
2. $x = \sin (y + a)$.

C. Using Eq. (22–25), find:

1. $\frac{d}{dx} (\cos^{-1} x)$.
2. $\frac{d}{dx} (\tan^{-1} x)$.
3. $\frac{d}{dx} (\sec^{-1} x)$.

22–6. Differentiation of Implicit Functions. If $y = f(x)$, we say that y is an *explicit function* of x. If, however, y is related to x by an expression of the type $\phi(x,y) = 0$, we say that y is an *implicit function* of x and that x is an implicit function of y. In the expression $xy^2 - 1 = 0$, x and y are implicit functions of each other. If we solve this equation for y, we obtain y as an explicit function of $x : y = \pm\sqrt{\frac{1}{x}}$; and if we solve the equation for x, we obtain x as an explicit function of $y : x = \frac{1}{y^2}\cdot$

It is possible to find $\frac{dy}{dx}$ from an implicit function of x and y without first solving explicitly for either variable. The implicit function is differentiated with respect to x giving rise to an equation in x, y, and $\frac{dy}{dx}\cdot$ This equation is then solved for $\frac{dy}{dx}\cdot$

Example. Find $\frac{dy}{dx}$; given $x^2 + y^2 = 10$.

Differentiating, we get

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = -\frac{x}{y}\cdot$$

By Eq. (22-27),

$$\frac{d\mathbf{V}}{d\theta} = j\mathbf{V},$$

and, by Eq. (22-8),

$$\frac{d\theta}{dt} = \frac{dt^2}{dt} = 2t.$$

Thus

$$\frac{d\mathbf{V}}{dt} = j2t\mathbf{V} = 2t\underline{/t^2 + \frac{\pi}{2}}.$$

Exercise 22-6

A. Find $\dfrac{d\mathbf{V}}{d\theta}$ for the following:

1. $\mathbf{V} = 5\underline{/\theta}$.
2. $\mathbf{V} = 1\underline{/\theta - \frac{\pi}{3}}$.

B. Find $\dfrac{d\mathbf{V}}{dt}$ for the following:

1. $\mathbf{V} = 2\underline{/\omega t}$.
2. $\mathbf{V} = A\underline{/m \sin \omega t}$.

22-8. Concerning Limits. In our dealings with limits we have used, and we shall continue to use, intuitive, rather than precise, arguments. In Secs. 22-4 and 22-5 we have deliberately avoided any attempt at providing the proofs, which involve considerations of limits (although in Sec. 22-3 we have indicated the basic method). This procedure is justified in a first study of communications mathematics inasmuch as rigorous treatments of theorems on limits are intricate, and their presentation may serve to cloud, rather than to clear, the significance of the end results.

For an introductory study of limits the interested student is referred to almost any standard text on calculus or, in particular, to the excellent treatment of G. H. Hardy, *Pure Mathematics* (Cambridge University Press, London, 1933).

Exercise 22-7

A. For a varying current to flow in an inductance, L, an emf of $e = L\frac{di}{dt}$ volts must be impressed across the inductance, where L is in henrys and $\frac{di}{dt}$ is in amperes per second. Find the applied emf's which will produce sinusoidal current of amplitude 2 milliamperes in an inductance of 0.005 henry at the following frequencies:

1. 100 cycles per second.
2. 10 kilocycles per second.
3. 3 megacycles per second.

B. The charging current varies with the applied potential across a capacitance in accordance with the relation $i = C\frac{de}{dt}$ amperes, where C is in farads and $\frac{de}{dt}$ is in volts per second. Find the charging current which results with a capacitance of 0.005 microfarad for sinusoidal applied potentials of amplitude 3 volts at the following frequencies:

1. 100 cycles per second.
2. 10 kilocycles per second.
3. 3 megacycles per second.

CHAPTER 23

APPLICATION OF DERIVATIVES

23–1. Maxima and Minima. If for $x = a$, a function $f(x)$ assumes a value $f(a)$ such that $f(a)$ is greater than values of the function in the immediate neighborhood of $x = a$, then we shall say that $f(a)$ is a *maximum.* If for $x = b$ a function $f(x)$ assumes a value $f(b)$ such that $f(b)$ is less than values of the function in the immediate neighborhood of $x = b$, then we shall say that $f(b)$ is a *minimum.* According to these definitions the function the graph of which is shown in Fig. 23–1 has maxima at A, B, and C, and minima at D, E, and F. We refer to A, B, and C as *maximum points,* and to D, E, and F as *minimum points.* Where a curve passes through a maximum point or a minimum point, the tangent to the curve at this point is horizontal (slope zero). Thus we see that a maximum or a minimum of a given function is associated with a zero value of the derivative of the function.

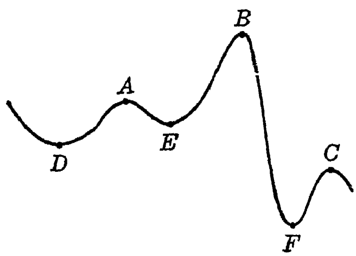


FIG. 23–1. Curve with maximum points at A, B, and C; minimum points at D, E, and F.

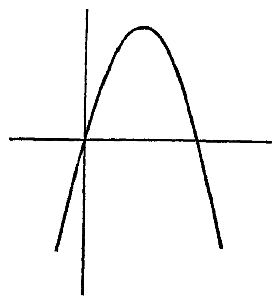


FIG. 23–2. Graph of $y = 4x - x^2$.

Let us consider the function $y = 4x - x^2$, the graph of which is shown in Fig. 23–2. To locate the maximum we may observe it directly on the graph, or we may proceed analytically as follows. The derivative of $4x - x^2$ is

$$\frac{dy}{dx} = 4 - 2x.$$

Equating the derivative to zero, we have

$$4 - 2x = 0,$$

the solution of which is

$$x = 2.$$

The maximum occurs, then, at $x = 2$. The value of the maximum, 6, is obtained by substitution of 2 for x in the equation $y = 4x - x^2$.

If instead of the relation $y = 4x - x^2$ we had considered $y = 4x + x^2$, we should have found a zero value for the derivative at $x = -2$, which in this instance corresponds to a minimum (Fig. 23-3). One of the simplest ways to ascertain whether a point corresponds to a maximum or to a minimum, without actually constructing a curve, is to test by substituting into the given function values of x which are just slightly less than, and just slightly greater than, that value of x which holds for the point in question. For example, if in $4x + x^2$ we substitute for x first -2 and then successively -2.1 and -1.9, we obtain first -4 and then -3.79 and -3.99, respectively. The value of the function corresponding to $x = -2$ is, thus, a minimum.

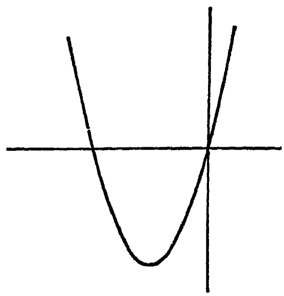


FIG. 23-3. Graph of $y = 4x + x^2$.

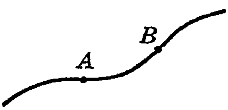


FIG. 23-4. Curve with inflection points at A and at B. At A the tangent to the curve is horizontal, and the derivative of the function is zero.

Unusual cases may arise in which a zero value of a derivative corresponds neither to a maximum point nor to a minimum point, but to an *inflection point*. An inflection point is a point which separates an upward concave portion of a curve ($\smile$) from a downward concave portion ($\frown$). Inflection points are shown at A and at B on the curve of Fig. 23-4. At the particular inflection point A the curve is of zero slope,* and hence a zero value of the derivative of the function is associated with the point A.

A zero value of the derivative of a function may thus indicate a maximum, a minimum, or an inflection point. Let us consider a function $f(x)$ for which the derivative vanishes at $x = a$. If there is an

* For brevity we shall henceforth speak of "the slope of the curve" as implying "the slope of the tangent to the curve."

inflection point at $x = a$ in the curve of the function, then this inflection point is manifested on examination of the function alone (without reference to the curve) in that $f(a)$ is either (1) greater than $f(a + h)$ and less than $f(a - h)$, or (2) less than $f(a + h)$ and greater than $f(a - h)$, where h is regarded here as a small positive quantity. (We shall examine the permissible magnitude of h later.) This is in contradistinction to the result in the case of a maximum or a minimum. For a maximum at $x = a$, $f(a)$ is greater than either $f(a + h)$ or $f(a - h)$. And for a minimum, $f(a)$ is less than either $f(a + h)$ or $f(a - h)$.

Let us refer to any point at which a curve possesses a horizontal tangent as a *critical point* and, in particular, let us refer to an inflection point at which a curve possesses a horizontal tangent as a *critical inflection point.*

The function $y = f(x)$ of Fig. 23–5 has critical points at $x = a$, at $x = b$, at $x = c$, and at $x = d$; a, b, c, and d are the values of x which satisfy the equation $\frac{d}{dx} f(x) = 0$. In testing the function $f(x)$ to determine the type of critical point at $x = b$ we compare the values of $f(b - h_1)$ and $f(b + h_2)$. Now by examination of the curve we see that in our choice of abscissas, $b - h_1$ and $b + h_2$, we are confined only to

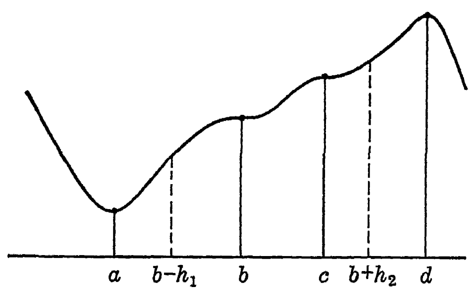


FIG. 23–5. Values of x within the interval $a \leq x \leq d$ are adequate for testing the critical point at $x = b$.

the intervals between a and b and between b and d; that is, we are confined only to within intervals of abscissas which are bounded by adjacent points of maximum or minimum. However, in making a test of a critical point, wherein, of course, we do not have the curve of the function available, we may not know beforehand the character of the adjacent critical points; in which case to be safe we restrict ourselves

to within intervals that are bounded by adjacent critical points. In the case of the function which is represented by the curve of Fig. 23-5 we should restrict ourselves to the intervals: $a < b - h_1 < b$ and $b < b + h_2 < c$.

In the foregoing illustration of the function $y = 4x + x^2$, wherein we checked the critical point at $x = -2$ by substituting for x first -2.1 and then -1.9 in the expression $4x + x^2$, we might equally well have substituted instead -10 and 0, respectively, thereby simplifying the computations. For $x = -10$, $4x + x^2$ becomes $-40 + 100 = 60$; and for $x = 0$, $4x + x^2$ becomes $0 + 0 = 0$. The value of the function for $x = -2$ (at the critical point) is $-8 + 4 = -4$. Both 60 and 0 are greater than -4, indicating the minimum at $x = -2$.

In communications work we are usually not interested in ascertaining the location of critical inflection points of curves. We are frequently interested in design questions pertaining to maximum and minimum concepts as, for example, minimum losses, minimum distortion, minimum cost, maximum output, and maximum efficiency. But only rarely are we concerned with problems involving turning points of trends that might be mathematically related to critical inflection points. However, because critical inflection points manifest themselves through zero values of the derivative, we are obliged to consider them at least sufficiently to be able to distinguish them from maxima and minima.

Exercise 23-1

Determine maxima and minima of the following functions.

1. $y = x^2 + 3$.
2. $y = -x^2 + 2x - 1$.
3. $y = x^3 - 3x$.
4. $y = x^3$.
5. $y = 2x^3 - 3x^2 - 12x$.
6. $y = \dfrac{x}{1 + x^2}$.
7. $y = 2x^3 + x^2 + 10x$.

23-2. Alternative Method of Testing for Maxima and Minima. The curve shown in Fig. 23-6 has a maximum at A, a minimum at C, and critical inflection points at B and D. At each of these points, A, B, C, and D, the slope of the curve is zero. To the left of A, as indicated by the rising character of the curve, the slope is positive; between A and B, as indicated by the falling character of the curve, the slope is negative; between B and C the slope is again negative; between C and D the slope is positive, and to the right of D the slope is again positive. This means that as x increases: at A the derivative changes in sign from positive to

negative; at B the derivative goes from negative to zero to negative again; at C the derivative changes in sign from negative to positive; and at D the derivative goes from positive to zero to positive again.

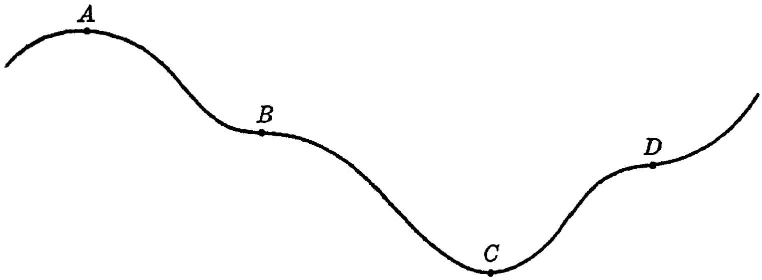


FIG. 23-6. Curve with a maximum, a minimum, and critical inflection points.

From these observations we deduce the following alternative test for a maximum or minimum at $x = a$ for a function $f(x)$ the derivative of which is zero at $x = a$: If the derivative changes in sign from positive to negative as x passes in an increasing sense through the value $x = a$, then the function is a maximum at $x = a$. If the derivative changes in sign from negative to positive as x passes in an increasing sense through the value $x = a$, then the function is a minimum at $x = a$.

This test may usually be simplified by a consideration of that function which represents the derivative of the derivative. This will be discussed in Sec. 23-4.

23-3. Higher Order Derivatives. As noted in Sec. 11-13 the derivative of a derivative is called a second derivative, and the derived curve of a derived curve is called a second derived curve. For a function $y = f(x)$ we may denote the first derivative by $y' = f'(x)$, and the second derivative by $y'' = f''(x)$. Then for any particular value of x, $x = a$: (1) the ordinate of the curve of $f(x)$ is equal to $f(a)$; (2) the slope of the tangent to the curve of $f(x)$ is equal to $f'(a)$; and (3) the rate at which the slope is changing is equal to $f''(a)$.

The notions of second derivative and second derived curve may be extended to include the *nth derivative* and the *nth derived curve.* Finding the successive derivatives of a function is known as *successive differentiation.* The nth derivative of a function $y = f(x)$ may be denoted by $y^{(n)}$, by $f^{(n)}(x)$, by $\dfrac{d^ny}{dx^n}$, or by $D_x^n(y)$.

Example. Find the first three derivatives of $y = x^4 + 5x + 2$.

First derivative: $y' = 4x^3 + 5$.
Second derivative: $y'' = 12x^2$.
Third derivative: $y''' = 24x$.

Exercise 23–2

Find the first three derivatives of each of the following functions.

1. $y = 2x^2$.
2. $y = x^3 - \frac{1}{2}x^2$.
3. $y = \epsilon^x$.
4. $y = \sin \theta$.
5. $y = \sin \omega t$.
6. $y = \dfrac{1}{x}$.

23–4. Geometrical Significance of Higher Order Derivatives. Curve 1 of Fig. 23–7 is concave upward at $x = a$. In other words, in the neighborhood of $x = a$ the slope is increasing as x increases. This is shown by the first derived curve (directly below curve 1), the ordinate of which is equal to the slope of the original curve at the corresponding abscissas in the neighborhood of $x = a$. The increasing nature of the first derived curve in the neighborhood of $x = a$ is, in turn, reflected in the positive ordinate of the second derived curve at $x = a$.

Curve 2 of Fig. 23–7 is concave downward at $x = a$. In other words, in the neighborhood of $x = a$ the slope is decreasing as x increases. Here the first derived curve is consequently decreasing in the neighborhood of $x = a$, and the ordinate of the second derived curve is negative at $x = a$.

Curves 1 and 2 of Fig. 23–7 have the same ordinate and the same slope at the abscissa $x = a$. But because of the different trends of the curves in the neighborhood of $x = a$, there are different ordinates of the second derived curves at $x = a$. Not only are the ordinates of the two second derived curves different at $x = a$, but the trends of the two second derived curves are different in the neighborhood of $x = a$ (in accordance with the different trends of the first derived curves); and, hence, the ordinates of the third derived curves differ at $x = a$. (See Exercise 23–3, Problems A2 and A3.) We conclude, then, that the nature of a curve in the neighborhood of any abscissa is reflected in the ordinates of the successive derived curves at this abscissa. We shall make use of this result in Sec. 24–1 in the construction of functions to fit experimental data.

An immediate practical consequence of these observations is that we

may express the alternative test for maxima and minima of Sec. 23-2 as follows:

Given a function $f(x)$ for which $f'(a) = 0$; if $f''(a)$ is positive, $f(a)$ is a minimum; and if $f''(a)$ is negative, $f(a)$ is a maximum.

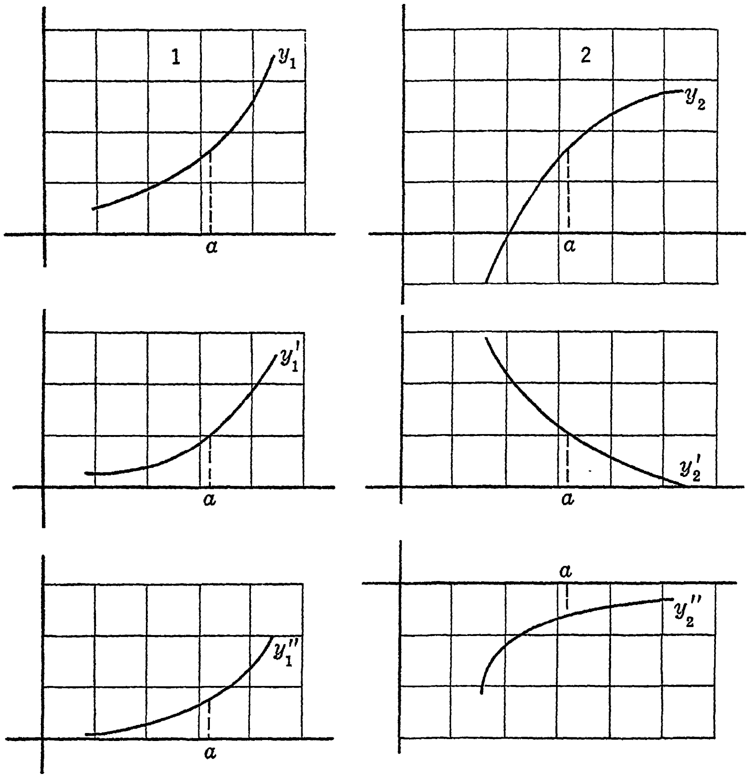


FIG. 23-7. Successive derived curves.

In case $f''(a)$ is zero we are obliged to examine the situation on both sides of $x = a$ as prescribed for the first derivative in Sec. 23-2, or as prescribed for the original function in Sec. 23-1.

Example. Examine the function $y = x^5 + 5x^4 + 5x^3 - 2 = 0$ for maxima and minima.

$$\begin{aligned} y' &= 5x^4 + 20x^3 + 15x^2 \\ &= 5x^2(x^2 + 4x + 3) \\ &= 5x^2(x+3)(x+1). \end{aligned}$$

The equation $y' = 0$ has solutions

$$x = 0,\ x = -3,\ x = -1.$$

These, then, are the critical values of x at which we wish to examine the second derivative. From

$$y' = 5x^4 + 20x^3 + 15x^2,$$

we obtain on differentiation

$$y'' = 20x^3 + 60x^2 + 30x.$$

Since we are interested only in determining the sign of y'' for particular values of x, we may equally well examine the sign of the function u, where

$$u = \frac{y''}{10} = 2x^3 + 6x^2 + 3x.$$

For $x = 0$, $u = 0$ and, hence, $y'' = 0$.
For $x = -3$,

$$u = -54 + 48 - 6 = -12,$$

and, hence, y'' is negative.
For $x = -1$,

$$u = -2 + 6 - 3 = 1,$$

and, hence, y'' is positive; $f(-3)$ is thus a maximum, and $f(-1)$ is a minimum. To determine the nature of the critical point at $x = 0$, let us evaluate $f(1)$ and $f(-1)$.

$$f(1) = 1 + 5 - 5 - 2 = -1,$$

and

$$f(-1) = -1 + 5 - 5 - 2 = -3.$$

Hence $x = 0$ corresponds to an inflection point.

Exercise 23-3

A. 1. Draw the third derived curve for curve 1 of Fig. 23-7.

2. With reference to Fig. 23-7 construct a curve y_3 which is such that at $x = a$: $y_3 = y_1$, $y_3' = y'$, $y_3'' = y''$, but $y_3''' \neq y_1'''$.

3. With reference to Fig. 23-7 construct a curve y_4 which is such that at $x = a$: $y_4 = y_2$, $y_4' = y_2'$, $y_4'' = y_2''$, but $y_4''' \neq y_2'''$.

B. Using the method of test suggested in Sec. 23-4, examine each of the functions of Exercise 23-1 for maxima and minima.

Exercise 23-4

1. The current which flows in a generator load is given by

$$I = \frac{E}{(R_g + jX_g) + (R_l + jX_l)}.$$

where E is the generator emf, $R_g + jX_g$ is the generator internal impedance, and $R_l + jX_l$ is the load impedance. The power developed in the load is

$$P = I^2R.$$

Show that maxmum power is absorbed in the load when the load impedance is the conjugate of the generator internal impedance. (Problem 5 of Exercise 8-4 is a special case of this problem for a resistive load.)

2. The magnitude of the current through an RLC series circuit is given by the equation

$$I = \frac{E}{\sqrt{R^2 + (X_L - X_C)^2}};$$

and the magnitude of the voltage across the capacitor is given by the equation

$$E_C = IX_C.$$

Show that if X_C is varied while R and L are fixed, then maximum voltage across the capacitor occurs when

$$X_C = X_L + \frac{R^2}{L}.$$

(Note that maximum voltage across the capacitor occurs for a value of X_C which is different from that for maximum current through the capacitor. The expression for I above is a maximum when the denominator is a minimum, that is, when $X_C = X_L$.)

CHAPTER 24

EMPIRICAL FORMULAS. EXPANSIONS OF FUNCTIONS

24–1. Empirical Formulas. A plot of experimentally observed current and voltage values for a metallic conductor is shown in Fig. 24–1. To within experimental error these points all lie on a line and, hence, we can at once devise a formula which describes the experimental relationship. Such a formula is $I = mE$, where m is the slope of the line. An equation, such as, $I = mE$, which is constructed to fit experimental data, is called an *empirical formula*.

The fit in the particular case illustrated — Ohm's Law — happens to be exceedingly good. It is to be emphasized, however, that Ohm's Law is not an exact, but an approximate, description of the observed facts. That the current in a conductor at a fixed temperature follows a relation which is only approximately, but not exactly, represented by $I = \frac{E}{R}$ is revealed by very accurate experiments. The approximation of the simple mathematical equation, $I = \frac{E}{R}$, to the course of the observed current and voltage is so good, however, that it is highly satisfactory to study circuits and to design new instruments on the basis of this empirical formula.

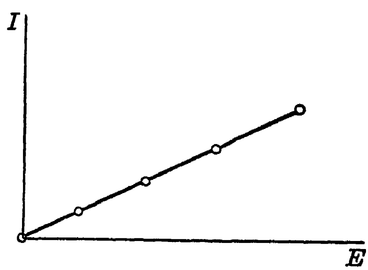


FIG. 24–1. Graph of relationship described by the empirical formula $I = mE$.

Likewise, we find it practical to devise mathematical equations which approximately describe many other observed relationships, for example, the relationship between the grid-to-cathode signal voltage and the varying component of the plate current in a radio vacuum tube. A plot of experimentally observed plate current-grid voltage values for a particular tube is shown in Fig. 24–2. The problem of finding a formula to fit the observed data is not as easy in this case as in the case of the metallic conductor, and it is not to be anticipated that any simple for-

mula will give an approximation which as nearly fits the experimental observations here as does the linear formula of Ohm's Law for the current-voltage relationships in a metallic conductor. Fortunately, however, in the problem at hand, as in many communications problems, a less exacting approximation is quite adequate for most analysis and design.

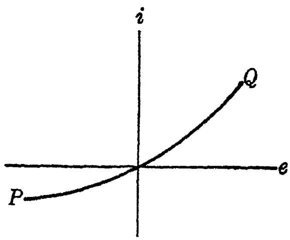


FIG. 24-2. Varying component of plate current as a function of signal voltage for a triode vacuum tube.

Let us consider the equation of the line AB in Fig. 24-3(a) as a first approximation to the equation of the experimental curve of Fig. 24-2, which is here designated as PQ. The line AB is arbitrarily selected to be tangent to the curve PQ at the origin. If we write $\phi(e)$ as the symbol for the unknown function which (of all the functions possessing derivatives at $e = 0$) best describes PQ, then the slope of AB is given by

$$m = \phi'(0),$$

where $\phi'(0)$ represents the derivative of $\phi(e)$ for $e = 0$. And the equation of AB is

$$i = me = \phi'(0)e.$$

Since the curve PQ is concave up, both to the left and to the right of the origin, this suggests that an improved approximation to PQ could be attained by adding the ordinates of a curve of equation $i = ke^2$ [Fig.

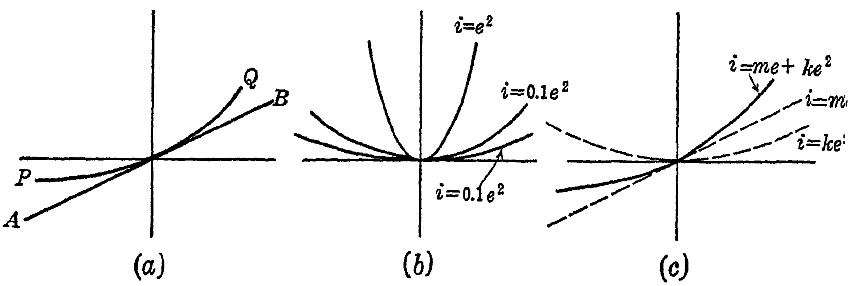


FIG. 24-3. Steps in the construction of a curve of known equation to approximate the experimental curve PQ in the neighborhood of the origin. (a) First approximation, AB, $i = me$; (b) various curves with equations of the form $i = ke^2$; (c) result of adding $i = me$ and $i = ke^2$.

24-3 (b) and (c)] to the ordinates of the line. Suppose we arbitrarily choose k so that the second derivative of ke^2, which is $2k$, is equal to the second derivative of $\phi(e)$. The resulting function, $i = me + ke^2$, as we shall show shortly, is such that: (1) it is equal to $\phi(e)$ at $e = 0$; (2) its first derivative is equal to the first deiivative of $\phi(e)$ at $e = 0$; and (3) its second derivative is equal to the second derivative of $\phi(e)$ at $e = 0$. The significance of the remark (1) in the preceding sentence is that the empirical and the experimental curves coincide at $e = 0$; the significance of (2) is that the slope of the empirical curve is the same as that of the experimental curve at the point of coincidence; and the significance of (3) is that the general tendencies of the two curves are the same in the neighborhood of the point of coincidence.* In the neighborhood of $e = 0$, then, our constructed equation, or empirical formula, provides a working relation which approximately represents the observed facts.

The equalities which were stated in the preceding paragraph are readily demonstrated. Since $m = \phi'(0)$ and $2k = \phi''(0)$, we may write $i = me + ke^2$ as

$$i = \phi'(0)e + \tfrac{1}{2}\,\phi''(0)e^2, \tag{24-1}$$

from which we obtain on successive differentiation

$$i' = \phi'(0) + \phi''(0)e \tag{24-2}$$

and

$$i'' = \phi''(0). \tag{24-3}$$

For $e = 0$, Eqs. (24-1), (24-2), and (24-3) become, respectively,

$$i = 0 + 0 = \phi(0),$$

$$i' = \phi'(0) + 0 = \phi'(0),$$

and

$$i'' = \phi''(0).$$

An improved approximation is provided by an empirical function $i = me + ke^2 + le^3$, wherein $l = \frac{1}{6}\,\phi'''(0)$ (Exercise 24-1). For much vacuum tube work, however, it is found that a formula of the type $i = me + ke^2$ is satisfactory, and that the increase in accuracy gained by the longer formula, $i = me + ke^2 + le^3$, does not warrant the increased labor of the computations.

* Refer to Sec. 23-4 on the Geometrical Significance of Higher Order Derivatives.

Exercise 24-1

In Sec. 24-1 the function $i = me + ke^2 + le^3$ is constructed such that, for $e = 0$,

$$\begin{aligned} i &= \phi(e), \\ m &= \phi'(e), \\ k &= \tfrac{1}{2}\phi''(e), \\ l &= \tfrac{1}{6}\phi'''(e). \end{aligned}$$

Show that for $e = 0$, not only are the functions i and $\phi(e)$ equal, but the first three derivatives of i are respectively equal to the first three derivatives of $\phi(e)$.

24-2. Expansion of a Function. Just as in Sec. 24-1, a function $\phi(e)$ was approximated in the neighborhood of $e = 0$ by a two- or three-term constructed polynomial, so any function $f(x)$ all of whose first n derivatives exist at $x = 0$, may be approximated in the neighborhood of $x = 0$ by a polynomial of the form

$$y = A_0x^0 + A_1x^1 + A_2x^2 + A_3x^3 + A_4x^4 + \cdots + A_nx^n,^* \qquad (24\text{-}4)$$

where the number of terms to be employed is determined by the desired degree of accuracy. The coefficients in Eq. (24-4) are determined from the following relations which we impose for the particular condition of $x = 0$:

$$\begin{aligned} y &= f(x), \\ y' &= f'(x), \\ y'' &= f''(x), \\ y''' &= f'''(x), \\ y'''' &= f''''(x), \qquad (24\text{-}5) \\ &\cdot \\ &\cdot \\ &\cdot \\ y^{(n)} &= f^{(n)}(x). \end{aligned}$$

* For $x \neq 0$, $x^0 = 1$. (See Problem A4 of Exercise 6-3.) For $x = 0$ in Eq. (24-4) we might formally define $x^0 = 1$ just to permit us to write all the terms of y in the uniform manner of Eq. (24-4).

By successive differentiation of y in Eq. (24-4), we get

$$\begin{aligned}
y' &= A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots + nA_nx^{n-1},\\
y'' &= 2A_2 + 3\cdot 2A_3x + 4\cdot 3A_4x^2 + \cdots + n(n-1)\,A_nx^{n-2},\\
y''' &= 3\cdot 2A_3 + 4\cdot 3\cdot 2A_4x + \cdots + n\,(n-1)\,(n-2)\,A_nx^{n-3},\\
&\;\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad (24\text{–}6)\\
y^{(n)} &= n!\,A_n,
\end{aligned}$$

wherein the symbol $n!$ (read: "n factorial") means the continued product $1\cdot 2\cdot 3\cdots n$.

Then from Eqs. (24-4), (24-5), and (24-6) we have at $x = 0$:

$$\begin{aligned}
A_0 &= f(0),\\
A_1 &= f'(0),\\
2A_2 &= f''(0),\\
3\cdot 2A_3 &= f'''(0),\\
4\cdot 3\cdot 2A_4 &= f''''(0), \qquad (24\text{–}7)\\
&\;\;\vdots\\
n!\,A_n &= f^n(0).
\end{aligned}$$

We state, then, that a function, $f(x)$, may be represented for certain values of x with any desired degree of accuracy by a polynomial:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \qquad (24\text{–}8)$$

By definition we take $0! = 1$, which permits us, if we wish, to write Eq. (24-8) in what we shall call the *homogeneous form:*

$$f(x) \approx \frac{f^0(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \qquad (24\text{–}9)$$

Constructing the polynomial of Eq. (24-8) or Eq. (24-9) for any given function is known as *expanding the function in a power series.*

Example 1. Expand $\sin x$.

$$\begin{aligned} f(x) &= \sin x, \\ f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f'''(x) &= -\cos x, \\ f''''(x) &= \sin x, \\ f^{(5)}(x) &= \cos x. \end{aligned}$$

For $x = 0$, we have

$$\begin{aligned} f(0) &= \sin 0 = 0, \\ f'(0) &= \cos 0 = 1, \\ f''(0) &= -\sin 0 = 0, \\ f'''(0) &= -\cos 0 = -1, \\ f''''(0) &= \sin 0 = 0, \\ f^{(5)}(0) &= \cos 0 = 1. \end{aligned}$$

Then, by Eq. (24-8),

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Example 2. Expand $(1 + x)^m$.

$$\begin{aligned} f(x) &= (1 + x)^m, \\ f'(x) &= m(1 + x)^{m-1}, \\ f''(x) &= m(m - 1)(1 + x)^{m-2}, \\ f'''(x) &= m(m - 1)(m - 2)(1 + x)^{m-3}, \\ f''''(x) &= m(m - 1)(m - 2)(m - 3)(1 + x)^{m-4}; \end{aligned}$$

$$\begin{aligned} f(0) &= 1, \\ f'(0) &= m, \\ f''(0) &= m(m - 1), \\ f'''(0) &= m(m - 1)(m - 2), \\ f''''(0) &= m(m - 1)(m - 2)(m - 3). \end{aligned}$$

By Eq. (24-8),

$$(1 + x)^m \approx 1 + mx + \frac{m(m - 1)}{2!}x^2 + \frac{m(m - 1)(m - 2)}{3!}x^3 + \frac{m(m - 1)(m - 2)(m - 3)}{4!}x^4. \quad (24\text{-}10)$$

Exercise 24-2

1. Show that the expansion of cos x is

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

2. Show that the expansion of ϵ^x is

$$\epsilon^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

3. Express in homogeneous form the results of Example 1 and of Example 2 in Sec. 24-2 and the results of Problem 1 and of Problem 2 in this exercise.

24-3. Approximate Evaluations with the Binomial Expansion. By putting $x = \frac{1}{m}$ in Eq. (24-10), we obtain

$$\left(1 + \frac{1}{m}\right)^m \approx 1 + 1 + \frac{1}{2!}\frac{(m-1)}{m} + \frac{1}{3!}\frac{(m-1)(m-2)}{m^2} + \frac{1}{4!}\frac{(m-1)(m-2)(m-3)}{m^3} \qquad (24\text{-}11)$$

This is then an approximation expansion for the number ϵ which, we recall, is defined by the equation

$$\epsilon = \underset{m \to \infty}{\text{limit}} \left(1 + \frac{1}{m}\right)^m.$$

If we let m be a million, we have $m - 1$, $m - 2$, $m - 3$ all very nearly equal to m (to within a few parts in a million), and Eq. (24-11) becomes

$$\epsilon \approx \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}. \qquad (24\text{-}12)$$

By including more terms, Eq. (24-12) may be employed to evaluate ϵ to any desired accuracy. (See Exercise 24-3 A.)

When $|x| \ll 1$, x^2 is usually negligible as compared with x. Thus, for $x = 0.01$, $x^2 = 0.0001$; for $x = 0.001$, $x^2 = 0.000001$. If, then, $|x| \ll 1$, while at the same time $|m|$ is of about the order of magnitude of 1, or less, Eq. (24-10) may be written as

$$(1 + x)^m \approx 1 + mx. \qquad (24\text{-}13)$$

Eq. (24-13) is a useful approximation for simplification or evaluation of certain expressions. The following examples illustrate its application.

Example 1. Simplify $\sqrt{1-a}$, where $a \ll 1$.
By Eq. (24–13),

$$\sqrt{1-a} = (1-a)^{\frac{1}{2}} = 1 - \frac{a}{2}.$$

Example 2. Evaluate $\dfrac{1}{\sqrt{100.4}}$.
By Eq. (24–13),

$$\frac{1}{\sqrt{100.4}} = \frac{1}{\sqrt{100+0.4}} = \frac{1}{10\sqrt{1+0.004}} = 0.1(1+0.004)^{-\frac{1}{2}}$$
$$\approx 0.1(1-0.002) = 0.1(0.998) = 0.0998$$

Some special cases of Eq. (24–13) are listed below.

$$\frac{1}{1+x} \approx 1 - x.$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x.$$

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x.$$

$$\sqrt[3]{1+x} \approx 1 + \frac{3}{2}x.$$

$$\frac{1}{\sqrt[3]{1+x}} \approx 1 - \frac{3}{2}x.$$

Exercise 24–3

A. Obtain ϵ correct to five decimal places by using ten terms of the series of Eq. (24–12). Thus:

$$1.000000 + 1.000000 + 0.500000 + 0.166667 + \cdots.$$

B. Using Eq. (24–13), evaluate:

1. $\dfrac{1}{1.003}$.
2. $\dfrac{1}{10.03}$.
3. $\dfrac{1}{0.998}$.
4. $\dfrac{1}{9.96}$.
5. $\dfrac{1}{R^2+X^2}$, where $R \ll X$.

24-4. Taylor's Series. If in constructing a power series which shall approximate a given function, we choose to fit our constructed function to the given function in the neighborhood of $x = a$, instead of in the neighborhood of $x = 0$, we should then obtain by a process analogous to that of Sec. 24-2 (providing the first n derivatives of $f(x)$ exist at $x = a$):

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (24\text{-}14)$$

Eq. (24-12) is known as *Taylor's Series.*

By replacing x with $x + a$ in Eq. (24-14), we get another form of Taylor's Series:

$$f(x+a) \approx f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \cdots + \frac{f^{(n)}(a)}{n!}x^n. \quad (24\text{-}15)$$

We recognize Eq. (24-8) as that special form of Taylor's Series for which $a = 0$. Eq. (24-8) is usually referred to as *Maclaurin's Series.*

Example. Expand $\sin(a + x)$ in powers of x.

$$\begin{aligned} f(x) &= \sin x, \\ f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f'''(x) &= -\cos x, \\ f''''(x) &= \sin x. \end{aligned}$$

Then

$$\begin{aligned} f(a) &= \sin a, \\ f'(a) &= \cos a, \\ f''(a) &= -\sin a, \\ f'''(a) &= -\cos a, \\ f''''(a) &= \sin a. \end{aligned}$$

By Eq. (24-15),

$$\sin(a+x) \approx \sin a + x\cos a - \frac{x^2}{2!}\sin a - \frac{x^3}{3!}\cos a + \frac{x^4}{4!}\sin a.$$

Exercise 24–4

1. Show that cos $(a + x)$ expanded in powers of x is:

$$\cos(a + x) = \cos x - x \sin a - \frac{x^2}{2!}\cos a + \frac{x^3}{3!}\sin a + \frac{x^4}{4!}\cos a.$$

2. Show that ϵ^x expanded in powers of $x - 2$ is:

$$\epsilon^x \approx \epsilon^2 + \epsilon^2(x - 2) + \frac{\epsilon^2}{2!}(x - 2)^2 + \frac{\epsilon^2}{3!}(x - 2)^3.$$

24–5. Infinite Series. Series of indefinite numbers of terms — *infinite series* — may be constructed to represent exactly certain functions for particular values of the variable. By the value of an infinite series is meant that limit which is approached by the sum of the first n terms, S_n, as n is increased without bound. Any one of the approximate expansions which we have considered in Secs. 24–2 and 24–3 may be caused to represent exactly the corresponding function by considering the expansion as an infinite series. (The infinite series corresponding to Eq. (24–10) is valid only for $|x| < 1$.)

CHAPTER 25

FURTHER APPLICATIONS OF DERIVATIVES

25–1. Differentials. The symbol $\frac{dy}{dx}$ has been defined as a single entity, as the limit which is approached by the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches zero. For certain studies, however, it is convenient to ascribe interpretations to dx and dy individually.

Corresponding to a function $y = f(x)$ we define dx as equivalent to Δx, that is, as an arbitrary increment in x, and we define dy by the relation

$$dy = f'(x)\,dx, \tag{25–1}$$

where $f'(x)$ is the derivative of $f(x)$; dx is called the *differential of* x; dy is called the *differential of* y.*

Because of the manner in which the derivative occurs in Eq. (25–1) it is sometimes called the *differential coefficient.*

Exercise 25–1

Draw a curve representing an arbitrary function $y = f(x)$. From any point P on the curve lay off a horizontal line segment representing an increment of x, $\Delta x = dx$. Indicate the corresponding increments in y: Δy and dy. (It will be convenient to draw the tangent to the curve through P in order to establish the extent of dy.)

25–2. Formulas for Finding Differentials. Since by Eq. (25–1) the differential of any function is given by the product of the derivative and the differential of the independent variable, formulas for finding

* One can regard the differential as fundamental and establish the derivative as the ratio of differentials instead of the other way around as we have done here. For an interesting approach to derivatives through differentials, the student is referred to the not entirely vigorous but nevertheless practical monograph by Silvanus P. Thompson, *Calculus Made Easy* (The Macmillan Co., New York, 1936).

derivatives, which were listed in Sec. 22-4, may be converted into formulas for finding differentials. The results are listed below.

$$d(x) = dx. \tag{25-2}$$

$$d(kx) = kdx. \tag{25-3}$$

$$d(k) = 0. \tag{25-4}$$

$$d(x^n) = nx^{n-1}dx. \tag{25-5}$$

$$d(ku) = kdu. \tag{25-6}$$

$$d(u^n) = nu^{n-1}du \tag{25-7}$$

$$d(u + v) = du + dv. \tag{25-8}$$

$$d(uv) = udv + vdu. \tag{25-9}$$

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}. \tag{25-10}$$

$$d(\sin u) = \cos udu. \tag{25-11}$$

$$d(\cos u) = -\sin udu. \tag{25-12}$$

$$d(\tan u) = sec^2\, udu. \tag{25-13}$$

$$d(\cot)\, u = -\csc^2 udu. \tag{25-14}$$

$$d(\sec u) = \sec u \tan udu. \tag{25-15}$$

$$d(\csc u) = -\csc u \cot udu. \tag{25-16}$$

$$d(\epsilon^x) = \epsilon^x dx. \tag{25-17}$$

$$d(\epsilon^u) = \epsilon^u du. \tag{25-18}$$

$$d(a^u) = a^u(\ln a)du. \tag{25-19}$$

$$d \log_a{}^u = (\log_a \epsilon)\frac{du}{u}. \tag{25-20}$$

The use of the term *differentiation* is extended to include the process of finding differentials.

Applications of the above equations are given in the following examples.

Example 1. Differentiate $y = 2x^3 + 3x - 1$.

$$dy = (6x^2 + 3)dx.$$

Example 2. Differentiate $y = \sqrt{1 - x^2}$.
Write

$$1 - x^2 = u,$$

so that

$$du = -2xdx$$

and

$$y = u^{\frac{1}{2}}.$$

Then

$$dy = \tfrac{1}{2}u^{-\frac{1}{2}}\,du$$
$$= \frac{-2xdx}{2\sqrt{1 - x^2}} = -\frac{xdx}{\sqrt{1 - x^2}}.$$

Example 3. Differentiate $r = \sin\dfrac{1}{\alpha}$.
Write

$$\frac{1}{\alpha} = u,$$

so that

$$du = -\frac{1}{\alpha^2}\,d\alpha$$

and

$$r = \sin u.$$

Then

$$dr = \cos u du$$
$$= -\frac{1}{\alpha^2}\cos\left(\frac{1}{\alpha}\right)d\alpha.$$

Exercise 25

Differentiate, using differentials:

1. $y = 7x^4 - 2x^3 + x.$
2. $z = \sqrt{x^3 - 2x^2}.$
3. $y = \cos^2 x.$
4. $y = \sin^3(1 - x).$
5. $r = \dfrac{t + 1}{t^2 + 1}.$
6. $y = \dfrac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}}.$
7. $y = \epsilon^{ax}\cos bx.$

25-3. Partial Derivatives. Vacuum Tube Parameters. A function of two or more independent variables can be differentiated with respect to one of the variables if, during the operation, the other variables are held constant. The derivative obtained in this manner is called a *partial derivative.* For a function $z = f(x,y)$, where x and y are independent variables, the partial derivative of z with respect to x is indicated by either $\frac{\partial z}{\partial x}$ or $\left.\frac{dz}{dx}\right|_y$ · The subscript y in the latter expression for the partial derivative denotes " y held constant." The partial derivative of z with respect to x is indicated by either $\frac{\partial z}{\partial y}$ or $\left.\frac{dy}{dz}\right|_x$, the subscript x in the latter expression denoting " x held constant."

The plate current in a vacuum triode is given by a function of the form

$$i_b = f(e_b + \mu e_c),$$

where e_b is the plate voltage, e_c is the grid voltage, and μ (voltage amplification) is a factor which measures the relative effectiveness of grid and plate voltages in controlling the plate current. μ may be regarded as essentially constant over the range of voltages and currents with which we shall be concerned. The exact nature of the function which represents the current is not important for our purposes. However, it is approximately given by $A(e_b + \mu e_c)^n$, where A and n are design constants. e_b, being independent of e_c, is not affected by any change in e_c; hence, $\frac{de_b}{de_c} = 0$. And, likewise, $\frac{de_c}{de_b} = 0$. Then if for simplicity we write $i_b = f(v)$, where $v = e_b + \mu e_c$, we have

$$\begin{aligned} \left.\frac{di_b}{de_c}\right|_{e_b} &= \frac{di_b}{dv} \cdot \frac{dv}{de_c} \\ &= \frac{di_b}{dv} \cdot \mu; \end{aligned} \tag{25-21}$$

and

$$\begin{aligned} \left.\frac{di_b}{de_b}\right|_{e_c} &= \frac{di_b}{dv} \cdot \frac{dv}{de_b} \\ &= \frac{di_b}{dv} \cdot \end{aligned} \tag{25-22}$$

Division of Eq. (25-21) by Eq. (25-22) yields

$$\frac{\left.\dfrac{di_b}{de_c}\right|_{e_b}}{\left.\dfrac{di_b}{de_b}\right|_{e_c}} = \mu. \tag{25-23}$$

The numerator in the left expression in Eq. (25-23) is the transconductance, denoted by g_m:

$$g_m = \left.\frac{di_b}{de_c}\right|_{e_b}; \tag{25-24}$$

and the denominator in the same expression is the plate conductance, or reciprocal resistance, denoted by $\frac{1}{r_p}$:

$$\frac{1}{r_p} = \left.\frac{di_b}{de_b}\right|_{e_c}. \tag{25-25}$$

Hence, Eq. (25-23) may be written

$$g_m r_p = \mu, \tag{25-26}$$

a fundamental relation in vacuum tube studies.

μ may be expressed in terms of e_b and e_c. If we introduce the restriction that i_b is to remain constant, then

$$v = e_b + \mu e_c = C,$$

where C is a constant. e_b is no longer independent of e_c but is related to e_b through the equation

$$e_b = C - \mu e_c.$$

From the above expression,

$$\left.\frac{de_b}{de_c}\right|_{i_b} = -\mu,$$

or

$$\mu = -\left.\frac{de_b}{de_c}\right|_{i_b}. \tag{25-27}$$

Determination of the parameters μ, g_m, and r_p for a particular tube may be accomplished graphically from the characteristic curves of the

tube. By their definitive relations, μ [Eq. (25-27)] is equal to minus the slope of the e_b versus e_c (constant i_b) curve; g_m [Eq. (25-24)] is equal to the slope of the i_b versus e_c (constant e_b) curve; and r_p [Eq. (25-25)] is equal to the reciprocal of the slope of the i_b versus e_b (constant e_c) curve. All three parameters may be found from a single family of characteristics. The family of plate characteristics for a type 6J5

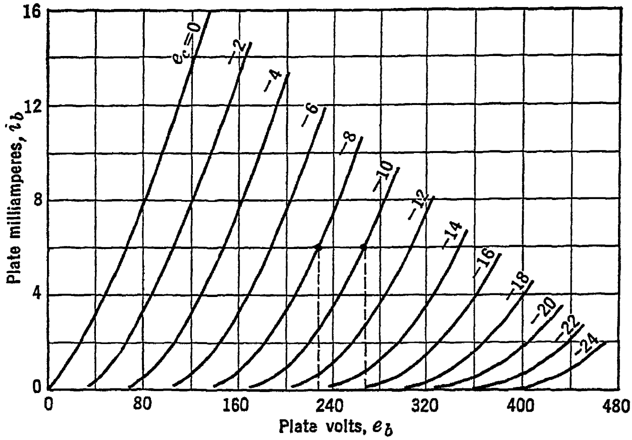


FIG. 25-1. Average plate characteristics for a type 6J5 triode.

triode is shown in Fig. 25-1. Here r_p is obtained from any particular curve (constant e_c) as the reciprocal of the slope of the curve. μ, being substantially constant, may be taken as approximately equal to $-\left.\dfrac{\Delta e_b}{\Delta e_c}\right|_{i_b}$, where Δe_c is the change in grid potential from one curve to the next, and where Δe_b is the difference between the plate potentials required to maintain the same plate current for the two grid potentials considered. g_m is then obtained from μ and r_p through Eq. (25-26). The slope of the curve for $e_c = -10$ through the point P in Fig. 25-1 is approximately 0.104 milliampere per volt. Hence, the plate resistance at P is equal to

$$r_p = \frac{1}{0.104} \text{ volts per milliampere}$$

$$= 9600 \text{ ohms.}$$

The difference in grid potential, Δe_c, between the curve through P and

the adjacent curve to the left, is

$$\Delta e_c = -10 - (-8) = -2 \text{ volts};$$

and the corresponding difference between plate potentials required to maintain constant i is

$$\Delta e_b = 266 - 227 = 39 \text{ volts.}$$

Hence,

$$\mu \approx -\left.\frac{\Delta e_b}{\Delta e_c}\right|_{i_b} = -\frac{39 \text{ volts}}{-2 \text{ volts}} = 20.$$

g_m at P follows from Eq. (25-26) as

$$g_m = \frac{\mu}{r_p} = \frac{20}{9600 \text{ ohms}}$$
$$= 2100 \text{ micromhos.}$$

Exercise 25-3

A. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions:

1. $z = 3xy + y^2$. **2.** $z = \epsilon^x \sin y$. **3.** $z = \dfrac{xy}{x + y}$.

B. For the type 6J5 tube (refer to Fig. 25-1), plot values of μ, g_m and r_p:

1. Versus e_b for $i_b = 6$ milliamperes.
2. Versus i_b for $e_b = 240$ volts.

C. A plot of i_b versus e_c with e_b constant is known as a transfer, or mutual, characteristic. From the curves of Fig. 25-1 plot a family of transfer characteristics including curves for $e_b = 120$, 160, 200, 240, and 280 volts. Now from this family of transfer characteristics graphically obtain values of μ, g_m, and r_p; and plot graphs of each of these parameters:

1. As a function of e_b, for $i_b = 4$ milliamperes.
2. As a function of i_b, for $e_b = 200$ volts.

D. An approximate expansion for the instantaneous a-c plate current expressed as a function of the instantaneous a-c grid voltage, for the plate potential e_b constant, is given by

$$i_p = A_1 e_g + A_2 e_g^2.$$

Note:

$$i_b = i_p + I_{bo}.$$

where I_{bo} is the (constant) quiescent value of the plate current determined by the operating conditions;
and

$$e_c = e_g + E_c,$$

where E_c is the (constant) average value of the grid voltage determined by the operating conditions.
Show that

$$A_1 = g_m = \frac{\mu}{r_p};$$

and that

$$A_2 = \frac{1}{2}\frac{\partial g_m}{\partial e_g} = -\frac{\mu}{2r_p^2}\frac{\partial r_p}{\partial e_g},$$

where μ is regarded as a constant and where each of the above expressions is to be evaluated at the quiescent operating point, that is, for $e_g = 0$.

CHAPTER 26

INTEGRATION AS INVERSE DIFFERENTIATION

26–1. Integration. *Integration* is the process of finding a function for which a given function is the derivative. We considered this problem graphically in Sec. 11–14 when we constructed integral curves from given curves. Now we approach the same problem analytically.

If a function $f'(x)$ is the derivative of a given function $f(x)$, then $f(x)$ is said to be an *integral of $f'(x)$ with respect to x.* The process of finding an integral of a function $f'(x)$ is denoted by the *integral sign* $\int$. Since

$$d(x^2) = 2x\,dx,$$

we write

$$\int 2x\,dx = x^2.$$

In the same manner, corresponding to

$$d \sin x = \cos x\,dx,$$

we write

$$\int \cos x\,dx = \sin x.$$

26–2. Constant of Integration. Inasmuch as the derivative of a constant is zero, we have the same derivative for the function $f(x)$ as for the function $f(x) + C$. For example,

$$d(x^2) = 2x\,dx,$$

and also

$$d(x^2 + 3) = 2x\,dx$$

and

$$d(x^2 - 5) = 2x\,dx.$$

This means, then, that $2x$ is the integral of an indefinite number of functions $x^2 + C$, where C is a constant (independent of x); or that, in

general, if $f'(x)$ is the derivative of $f(x)$, then

$$\int f'(x)\,dx = f(x) + C.$$

Since the constant C is indefinite, the expression $f(x) + C$ is called the *indefinite integral* of $f'(x)$ with respect to x.

26-3. The Integration Process. The integration process is the inverse of the differentiation process, and the finding of an integral function corresponding to a given function involves usually either the recognition of the given function as the result of some previously considered differentiation (as in the examples of the foregoing sections) or the transformation of the given function to a form which is recognized to be the result of a differentiation.

A list of standard forms of integrals can be compiled from the differentials which were given in Sec. 25-2. Such a list is given below.

$$\int k\,dx = kx + C. \tag{26-1}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ for } n \neq -1. \tag{26-2}$$

$$\int ku = k\int du. \tag{26-3}$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \text{ for } n \neq -1. \tag{26-4}$$

$$\int (du + dv) = \int du + \int dv. \tag{26-5}$$

$$\int u\,dv = uv - \int v\,du. \tag{26-6}$$

$$\int \frac{u\,dv - v\,du}{v^2} = \frac{u}{v} + C. \tag{26-7}$$

$$\int \cos u\,du = \sin u + C. \tag{26-8}$$

$$\int \sin u\,du = -\cos u + C. \tag{26-9}$$

$$\int \sec^2 u\,du = \tan u + C. \tag{26-10}$$

$$\int \csc^2 u\, du = -\cot u + C. \tag{26-11}$$

$$\int \sec u \tan u\, du = \sec u + C. \tag{26-12}$$

$$\int \csc u \cot u\, du = -\csc u + C. \tag{26-13}$$

$$\int \epsilon^x\, dx = \epsilon^x + C. \tag{26-14}$$

$$\int \epsilon^u\, du = \epsilon^u + C. \tag{26-15}$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C. \tag{26-16}$$

$$\int (\log_a \epsilon) \frac{du}{u} = \log_a u + C. \tag{26-17}$$

$$\int \frac{du}{u} = \ln u + C. \tag{26-18}$$

$$\int \tan u\, du = \ln (\sec u) + C. \tag{26-19}$$

$$\int \cot u\, du = \ln (\sin u) + C. \tag{26-20}$$

$$\int \sec u\, du = \ln (\sec u + \tan u) + C. \tag{26-21}$$

$$\int \csc u\, du = \ln (\csc u - \cot u) + C. \tag{26-22}$$

Eq. (26–18) is obtained from Eq. (26–17) on putting $a = \epsilon$.

We may prove Eq. (26–5) as follows.

By Eq. (25–8),

$$d(u + v) = du + dv.$$

Hence,

$$\int (du + dv) = u + v + C.$$

Further,

$$\int du = u + C_1,$$

and

$$\int dv = v + C_2,$$

so that

$$\int du + \int dv = u + v + C_1 + C_2.$$

On writing

$$C_1 + C_2 = C,$$

we have

$$\int (du + dv) = \int du + \int dv.$$

We may prove Eq. (26-6) as follows.
By Eq. (25-9),

$$d(uv) = u\,dv + v\,du.$$

Hence,

$$\int d(uv) = \int (u\,dv + v\,du).$$

Let us write, at least formally,

$$u\,dv = dw, \text{ and } v\,du = dz.$$

Then, by Eq. (26-5),

$$\int (u\,dv + v\,du) = \int (dw + dz) = \int dw + \int dz = \int u\,dv + \int v\,du.$$

Since

$$\int d(uv) = uv + C,$$

we have

$$uv + C = \int u\,dv + \int v\,du,$$

or

$$\int u\,dv = uv + C - \int v\,du.$$

Now, inasmuch as a constant of integration arises on performing the integration $\int v\,du$, we may incorporate C into this constant and write simply

$$\int u\,dv = uv - \int v\,du.$$

The proofs of Eq. (26–3) and of Eqs. (26–19) through (26–22) are left as exercises. (Part A of Exercise 26–1.) The other equations for integrals are obtained directly from the equations for differentials by regarding integration as the inverse process of differentiation.

The use of the equations for integrals is illustrated in the following examples.

Example 1. Find $\int 3\,dx$.

By Eq. (26–1),

$$\int 3\,dx = 3x + C.$$

Example 2. Find $\int (2x^2 - 5x + 1)\,dx$.

By Eq. (26–5),

$$\int (2x^2 - 5x + 1)\,dx = \int 2x^2\,dx - 5x\,dx + dx$$

$$= \int 2x^2\,dx + \int (-5)x\,dx + \int dx.$$

Then, by Eq. (26–3),

$$\int 2x^2\,dx = 2\int x^2\,dx,$$

and

$$\int (-5)x\,dx = -5\int x\,dx.$$

By Eq. (26–2) for $n = 0$, 1, and 2, respectively,

$$\int dx = x + C_1,$$

$$\int x\,dx = \frac{x^2}{2} + C_2,$$

and

$$\int x^2\,dx = \frac{x^3}{3} + C_3.$$

Thus,

$$\int (2x^2 - 5x + 1)\,dx = \frac{2x^3}{5} - \frac{5x^2}{2} + x + C$$

on writing

$$C = C_1 + C_2 + C_3.$$

Example 3. Find $\int \frac{dx}{\sqrt{1 + x}}$.

Put

$$u = 1 + x.$$

Then

$$du = dx,$$

and

$$\int \frac{dx}{\sqrt{1 + x}} = \int u^{-\frac{1}{2}}\,du.$$

By Eq. (26-4),

$$\int u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= 2\sqrt{1 + x} + C.$$

Example 4. Find $\int \frac{4x\,dx}{x^2 + 1}$.

Put

$$u = x^2 + 1.$$

Then

$$du = 2x\,dx,$$

and

$$\int \frac{4x\,dx}{x^2 + 1} = \int \frac{2du}{u} = 2 \ln u + C$$
$$= 2 \ln (x^2 + 1) + C.$$

Example 5. Find $\int \frac{\cos\theta\,d\theta}{1 + 2\sin\theta}$.

Put

$$u = 1 + 2\sin\theta.$$

Then

$$du = 2\cos\theta\,d\theta,$$

and

$$\int \frac{\cos\theta\, d\theta}{1 + 2\sin\theta} = \int \frac{\frac{1}{2}\, du}{u} = \frac{1}{2}\ln u + C$$

$$= \tfrac{1}{2}\ln(1 + 2\sin\theta) + C.$$

Example 6. Find $\int \sin^2\theta\, d\theta$.

$$\int \sin^2\theta\, d\theta = \int \frac{(1 - \cos 2\theta)}{2}\, d\,\theta = \int \frac{d\theta}{2} - \int \frac{(\cos 2\theta)\, d\theta}{2}.$$

To integrate $\dfrac{(\cos 2\theta)\, d\theta}{2}$ put

$$u = 2\theta.$$

Then

$$du = 2d\theta,$$

and

$$\int \frac{(\cos 2\theta)\, d\theta}{2} = \int \frac{\cos u\, du}{4} = -\frac{\sin u}{4} + C_1$$

$$= -\frac{\sin 2\theta}{4} + C_1.$$

Thus,

$$\int \sin^2\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C.$$

Example 7. Find $\int \sin\theta\cos\theta\, d\theta$.

Put

$$u = \sin\theta.$$

Then

$$du = \cos\theta\, d\theta,$$

and

$$\int \sin\theta\cos\theta\, d\theta = \int u\, du = \frac{u^2}{2} + C$$

$$= \frac{\sin^2\theta}{2} + C.$$

Example 8. Find $\int \sin^2 \theta \cos^3 \theta \, d\theta$,

Since

$$\cos^2 \theta = 1 - \sin^2 \theta,$$

$$\begin{aligned}
\int \sin^2 \theta \cos^3 \theta \, d\theta &= \int \sin^2 \theta \, (1 - \sin^2 \theta) \cos \theta \, d\theta \\
&= \int \sin^2 \theta \cos \theta \, d\theta - \int \sin^4 \theta \cos \theta \, d\theta \\
&= \frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + C.
\end{aligned}$$

Example 9. Find $\int \frac{dx}{\sqrt{a^2 + x^2}}$

Put

$$x = a \tan \theta.$$

Then

$$dx = a \sec^2 \theta \, d\theta,$$

and

$$\begin{aligned}
a^2 + x^2 &= a^2 + a^2 \tan^2 \theta \\
&= a^2 (1 + \tan^2 \theta) \\
&= a^2 \sec^2 \theta.
\end{aligned}$$

Thus,

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a \sec^2 \theta \, d\theta}{a \sec \theta} = \int \sec \theta \, d\theta = \ln (\sec \theta + \tan \theta) + C.$$

Now

$$x = a \tan \theta,$$

so that

$$\tan \theta = \frac{x}{a},$$

$$\sec \theta = \frac{\sqrt{a^2 + x^2}}{a},$$

and

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(\frac{\sqrt{a^2 + x^2} + x}{a} \right) + C.$$

Example 10. Find $\int x\epsilon^x \, dx$.

In anticipation of using Eq. (26-6), put

$$u = x, \quad \text{and} \quad dv = \epsilon^x \, dx.$$

Then

$$du = dx, \text{ and } \quad v = \int \epsilon^x \, dx = \epsilon^x + C_1.$$

By Eq. (26–6),

$$\int x\epsilon^x \, dx = x(\epsilon^x + C_1) - \int (\epsilon^x + C_1) \, dx$$
$$= x\epsilon^x + C_1 x - \epsilon^x - C_1 x + C.$$
$$= (x - 1)\epsilon^x + C.$$

It is to be noted in the above example that no term involving C_1 appears in the final expression. We can show that this is true in general in using Eq. (26–6). Let us consider any integral of the form $\int uf' \, dx$ where $\int f' \, dx = f + C_1$. If we write

$$dv = f' \, dx,$$

so that

$$v = f + C_1,$$

we have

$$uv = uf + C_1 u,$$

and

$$-\int v \, du = -\int (f \, du + C_1 \, du).$$

Then, on using Eq. (26–6), the term $-C_1 u$ which arises on integrating $-v \, du$ cancels the term $C_1 u$ which is part of the expression for uv and, hence, no term involving C_1 appears in the final expression. For this reason the constant of integration which is associated with the expression for v may be ignored.

Example 11. Find $\int x \sin x \, dx$.

To solve by Eq. (26–6), put

$$u = x, \quad \text{and } dv = \sin x \, dx.$$

Then

$$du = dx, \text{ and } \quad v = -\cos x.$$

By Eq. (26–6),

$$\int x \sin x = -x \cos x - \sin x + C.$$

Exercise 26–1

A. Show that:

1. $\int k\,du = k \int du.$

2. $\int \tan u\,du = \int \frac{\sin u\,du}{\cos u}$
$= -\ln(\cos u) + C + \ln(\sec u) + C.$

3. $\int \cot u\,du = \int \frac{\cos u\,du}{\sin u}$
$= \ln(\sin u) + C.$

4. $\int \sec u\,du = \int \left(\frac{\sec u + \tan u}{\sec u + \tan u}\right) \sec u\,du$
$= \ln(\sec u + \tan u) + C.$

5. $\int \csc u\,du = \int \left(\frac{\csc u - \cot u}{\csc u - \cot u}\right) \csc u\,du$
$= \ln(\csc u - \cot u) + C.$

B. In each of Problems 1 through 11 following, compare the illustrated example of the same number in Sec. 26–1. Regard m and n as any positive numbers except for the restriction that, where m and n appear in the same integral, $m \neq n$. Develop from the integral relations of Sec. 26–3, and verify by differentiating the result in each case:

1. $\int \frac{dx}{2} = \frac{x}{2} + C.$

2. $\int (4x^5 - 2x^3)\,dx = \frac{2x^6}{3} - \frac{x^4}{2} + C.$

3. $\int \frac{dx}{\sqrt{a - x}} = -2\sqrt{a - x} + C.$

4. $\int \frac{2x\,dx}{3 - x^2} = -\ln(3 - x^2) + C.$

5. $\int \cos 2x \sin 2x\,dx = \frac{\sin^2 2x}{4} + C.$

6. $\int \cos^2 \theta\,d\theta = \frac{}{2} + \frac{\sin 2\theta}{4} + C.$

7. $\int \frac{\sec^2 \theta\,d\theta}{1 + \tan \theta} = \ln(1 + \tan \theta) + C.$

8. $\int \sin^3 \theta \, d\theta = -\frac{1}{3} \cos \theta \, (\sin^2 \theta + 2) + C.$

9. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C.$ (Put $u = a \sin \theta$.)

10. $\int x^2 \epsilon^{ax} \, dx = \frac{x^2 \epsilon^{ax}}{a} - \frac{2\epsilon^{ax}}{a^3} (ax - 1) + C.$

11. $\int x^2 \sin x \, dx = 2x \sin x + (2 - x^2) \cos x + C.$

12. $\int \sin 2x \, dx = -\frac{\cos 2x}{2} + C.$

13. $\int \cos 3x \, dx = \frac{\sin 3x}{3} + C.$

14. $\int \sin (1 - m)\alpha \, d\alpha = \frac{1}{m - 1} \cos (1 - m)\alpha + C.$

15. $\int \sin mx \sin nx \, dx = \frac{\sin (m - n)x}{2(m - n)} - \frac{\sin (m + n)x}{2(m + n)} + C.$

Use the identity of Eq. 19–9:

$$\sin mx \sin nx = \frac{\cos (m - n)x - \cos (m + n)x}{2}$$

16. $\int \sin mx \cos nx \, dx = -\frac{\cos (m - n)x}{2(m - n)} - \frac{\cos (m + n)x}{2(m + n)} + C.$

17. $\int \cos mx \cos nx \, dx = \frac{\sin (m - n)x}{2(m - n)} + \frac{\sin (m + n)x}{2(m + n)} + C.$

18. $\int \sin^2 nx \, dx = \frac{1}{2n} (nx - \sin nx \cos nx) + C.$

19. $\int \cos^2 nx \, dx = \frac{1}{2n} (nx + \sin nx \cos nx) + C.$

20. $\int \sqrt{a^2 - x^2} \, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$

21. $\int \epsilon^{-x} \sin x \, dx = -\frac{\epsilon^{-x}}{2} (\sin x + \cos x) + C.$

22. $\int \epsilon^{nx} \sin mx \, dx = \epsilon^{nx} \left(\frac{n \sin mx - m \cos mx}{m^2 + n^2} \right) + C.$

23. $\int \epsilon^{nx} \cos mx \, dx = \epsilon^{nx} \left(\frac{n \cos mx + m \sin mx}{m^2 + n^2} \right) + C.$

26-4. Integral Tables. Comprehensive tables of standard forms for integrals are available, such as *A Short Table of Integrals,* by B. O. Peirce (Ginn and Company, Boston, 1929). Many of the commonly occurring integrable functions are listed directly in an integral table. Many other functions, which do not appear in a table, are readily reducible to forms which do appear in a table by means of appropriate transformations similar to those employed in Sec. 26-3.

26-5. Determination of the Constant of Integration. The constant of integration may be determined by a condition which relates the integral and the independent variable. Thus, if in Example 2 of Sec. 26-3 it had been stipulated that the integral should have the value $\frac{7}{6}$ when $x = 1$, we should have had, together with the equation

$$\int (2x^2 - 5x + 1)\,dx = \frac{2x^3}{3} - \frac{5x^2}{2} + x + C,$$

the condition

$$\tfrac{7}{6} = \tfrac{2}{3} - \tfrac{5}{2} + 1 + C;$$

and this condition imposed on C requires that

$$C = 2.$$

Exercise 26-2

A. Find the function which is such that:

1. Its first derivative is $x + 2$, and the function is equal to 2 when $x = 4$.
2. Its first derivative is $2 \sin \theta \cos \theta + \cos \theta$, and the function is equal to 3 when $\theta = \dfrac{\pi}{4}$.

B. An object has an acceleration of constant magnitude a. (Acceleration is the time rate of change of velocity; $a = \dfrac{dv}{dt}$.) If time is reckoned from the moment at which the instantaneous velocity of the object is v_0:

1. Show that the velocity of the object at any time t is given by

$$v = v_0 + at.$$

2. Show that the distance s which the object travels during this time is given by

$$s = v_0 t + \frac{at^2}{2}.$$

C. In a cathode ray tube, as illustrated in Fig. 26-1, an electron entering the electric field between the horizontal plates (here shown edgewise) experiences a constant acceleration upward during the time it remains within the

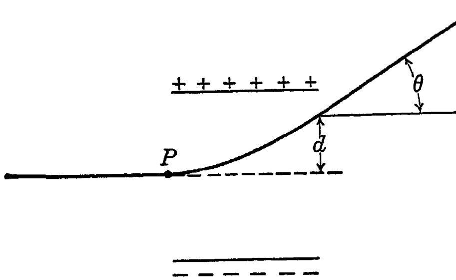


FIG. 26-1. Trajectory of electron in cathode ray tube.

plates. The effect of gravity on the electron is negligible. And the horizontal component of the velocity of the electron is essentially uniform throughout the path shown.

1. Set up horizontal and vertical coordinate axes through the point P of Fig. 26-1, and demonstrate that the trajectory of the electron within the plates may be represented by an equation of the form

$$y = kx^2.$$

(Suggestion:

$$\frac{dx}{dt} = C_1;$$

hence,

$$x = C_1t + C_2.$$

$$\frac{d^2y}{dt^2} = C_2;$$

hence,

$$\frac{dy}{dt} = C_2t + C_3.)$$

2. If the electron enters the field horizontally with a velocity of 10^9 centimeters per second, if the plates are 1 centimeter long, and if the potential between the plates is such as to cause an upward acceleration of the electron of 10^{18} centimeters per second per second, find d, the total deflection which the electron experiences within the plates, and θ, the angle with the horizontal at which the electron emerges. (Differentiate the expression which represents the trajectory. The derivative is

the slope of the trajectory. Evaluate the derivative at the point corresponding to $x = 1$ centimeter.)

26-6. Growth and Decay. Many problems in growth and decay, which are of considerable importance in electrical phenomena, are treated by methods which we now have at our disposal. Let us first investigate the growth and decay of current in an *RL* circuit. The equation of the circuit is

$$L\frac{di}{dt} + Ri = E. \tag{26-23}$$

The equation may be rewritten as

$$\frac{di}{i - \dfrac{E}{R}} = -\frac{R}{L}\,dt,$$

in which form it may be integrated to yield

$$\ln\left(i - \frac{E}{R}\right) = -\frac{Rt}{L} + C;$$

that is,

$$i - \frac{E}{R} = \epsilon^{-\frac{R}{L}t+C}.$$

$$= A\epsilon^{-\frac{R}{L}t}, \tag{26-24}$$

where

$$A = \epsilon^C.$$

If $i = 0$ when $t = 0$, as is the situation at the instant of closing a switch which establishes the circuit, then, by Eq. (26-24),

$$-\frac{E}{R} = A,$$

so that Eq. (26-24) becomes

$$i - \frac{E}{R} = -\frac{E}{R}\epsilon^{-\frac{R}{L}t},$$

or

$$i = \frac{E}{R}[1 - \epsilon^{-\frac{R}{L}t}]. \tag{26-25}$$

Eq. (26-25) is plotted for different values of the ratio $\frac{R}{L}$ in Fig. 26-2. In each case the current approaches the steady state value $\frac{E}{R}$ given by Ohm's Law.

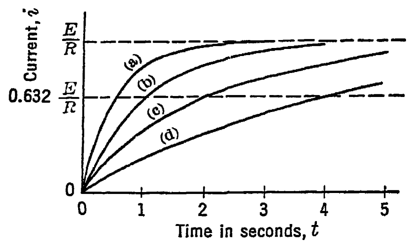


FIG. 26-2. Rise of current in an RL circuit.
(a) $\frac{L}{R} = \frac{1}{2}$; (b) $\frac{L}{R} = 1$; (c) $\frac{L}{R} = 2$; (d) $\frac{L}{R} = 4$.

The exponential, $\epsilon^{-\frac{R}{L}t}$, in Eq. (26-25) becomes simply ϵ^{-1}, or $\frac{1}{\epsilon}$, if we put

$$1 = \frac{R}{L}t$$

or

$$t = \frac{L}{R}.$$

Thus, we see that the current reaches a fraction $\left(1 - \frac{1}{\epsilon}\right)$, that is, 0.632, of its final value in the time $\frac{L}{R}$. This ratio $\frac{L}{R}$, called the time constant of the circuit, is indicative of the manner in which the current rises in the circuit. It would be futile to attempt to characterize an inductive circuit by the time it takes for the current to reach the steady state value $\frac{E}{R}$. (Why?)

To study the situation following the removal, by shorting out, of the applied potential in an RL circuit, we put $E = 0$ and take $i = i_0$ at time

$t = 0$. Then Eq. (26-24) becomes

$$i = i_0 \epsilon^{-\frac{R}{L}t}. \tag{26-26}$$

The current decays exponentially as shown in Fig. 26-3, falling to $\frac{1}{\epsilon}$, that is, 0.368, of its initial value in $\frac{L}{R}$ seconds.

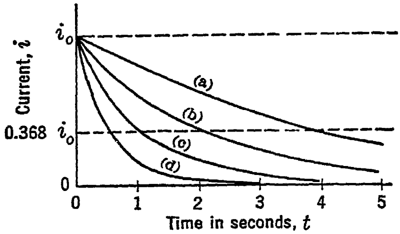


FIG. 26-3. Decay of current in an RL circuit. (a) $\frac{L}{R} = \frac{1}{2}$; (b) $\frac{L}{R} = 1$; (c) $\frac{L}{R} = 2$; (d) $\frac{L}{R} = 4$.

Exercise 26-3

A. In an RL circuit, of time constant 0.010 reciprocal second, how long after the application of a steady emf does the current reach:

1. 0.50 of its steady state value?
2. 0.90 of its steady state value?
3. 0.99 of its steady state value?

B. In an RL circuit, of resistance 120 ohms and inductance 0.50 henry, how long after removal (by shorting) of the emf does the current reach:

1. 0.10 of its initial value?
2. 0.0010 of its initial value?

C. The equation of an RC circuit is

$$R\frac{dq}{dt} + \frac{q}{C} = E.$$

1. Show that the charge on the condenser rises in accordance with the relation

$$q = CE[1 - \epsilon^{-\frac{1}{RC}t}],$$

where t is measured from the instant of closing the circuit.

2. If at the instant of applying the emf E (time $t = 0$) there is already a charge q_0 on the plates, show that

$$q = CE + (q_0 - CE)\epsilon^{-\frac{R}{C}t}.$$

D. In the circuit of Fig. 26-4 the potential across the RC circuit consisting of R_2 and C is $E - i_1R_1$. If by means of an auxiliary circuit (not shown) the current i_1 is controlled to be of square wave form, sketch the approximate wave shape of:

1. The charge on C.
2. The current i_2.
3. The voltage i_2R_2 cross R_2.

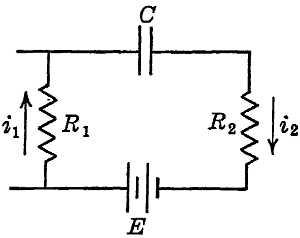


FIG. 26-4. RC circuit.

26-7. The Definite Integral. We are frequently interested in the difference between the values of an integral for two particular epochs. The integral $\int 2x\,dx$ is equal to $x^2 + C$. For $x = 4$ this is $16 + C$, and for $x = 1$ it is $1 + C$. The difference between these values, $16 + C$ and $1 + C$, is $16 - 1$ or 15. The difference between the values of an indefinite integral of $f(x)$ with respect to x for $x = a$ and for $x = b$ we call the *definite integral of $f(x)$ with respect to x between the limits of a and b.* If

$$\int f(x)\,dx = F(x) + C,$$

then we express the definite integral in writing by:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a).$$

It is to be observed: (1) that the constant of integration does not appear in the definite integral, and (2) that the definite integral is a function not of x but of the upper and lower limits, a and b. In accordance with (2) the integrals $\int_a^b f(x)\,dx$ and $\int_a^b f(\theta)\,d\theta$ are equivalent.

Example 1. Evaluate $\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta$.

$$\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = \sin\theta\Big]_0^{\frac{\pi}{2}} = 1 - 0 = 1.$$

Example 2. Evaluate $\int_0^1 \frac{dx}{a+x}$.

$$\int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big]_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Example 3. Evaluate $\int_0^a \frac{dx}{\sqrt{a^2+x^2}}$

(Compare Example 9 of Sec. 26-3.)

$$\int_0^a \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(\frac{\sqrt{a^2+x^2}+x}{a}\right)\Big]_0^a = \ln\left(\frac{\sqrt{2}a+a}{a}\right) - \ln\left(\frac{a}{a}\right)$$
$$= \ln(\sqrt{2}+1) = \ln 2.414.$$

When a change of variable is introduced to facilitate integration, it is sometimes advantageous to alter the limits of integration in accordance with the new variable instead of restoring the original variable in the expression for the integrated function. Thus, we might solve the problem of Example 3 above as follows: Referring to Example 9 of Sec. 26-3, we have

$$\int_{x=0}^{x=0} \frac{dx}{a^2+x^2} = \ln\left(\sec\theta + \tan\theta\right)\Big]_{\theta=\theta_1}^{\theta=\theta_2}.$$

θ_1 is the value of θ which obtains when $x = 0$, and θ_2 is the value of θ which obtains when $x = a$. From the transformation equation used in this problem,

$$x = a\tan\theta,$$

we have

$$\theta = \tan^{-1}\frac{x}{a}.$$

Hence, for $x = 0$, $\theta = 0$; and for $x = a$, $\theta = \frac{\pi}{4}$. Thus, the integral becomes

$$\ln(\sec\theta + \tan\theta)\Big]_{\theta=0}^{\theta=\frac{\pi}{4}} = \ln(\sqrt{2}+1) - \ln(1+0)$$
$$= \ln(\sqrt{2}+1) = \ln 2.414.$$

Exercise 26-4

Show that:

1. $\int_{-1}^{1} (2x+1)\,dx = 2.$

2. $\int_0^1 \frac{dx}{x+a} = \ln\left(\frac{1+a}{a}\right).$

3. $\displaystyle\int_0^a b\epsilon^{-x}\,dx = b(1 - \epsilon^{-a}).$

4. $\displaystyle\int_0^a \frac{dx}{a^2 + x^2} = \frac{\pi}{4a}.$

5. $\displaystyle\int_0^a \frac{a\,dx}{\sqrt{a^2 - x^2}} = \frac{\pi\,a}{2}.$

6. $\displaystyle\int_k^{k+2\pi} \sin nx\,dx = 0.$

7. $\displaystyle\int_k^{k+2\pi} \cos nx\,dx = 0.$

8. $\displaystyle\int_k^{k+2\pi} \sin^2 nx\,dx = \pi.$

9. $\displaystyle\int_k^{k+2\pi} \cos^2 nx\,dx = \pi.$

10. $\displaystyle\int_k^{k+2\pi} \sin mx \cos nx\,dx = 0.$

11. $\displaystyle\int_k^{k+2\pi} \sin mx \sin nx\,dx = 0$ for $m \neq n$.

12. $\displaystyle\int_k^{k+2\pi} \cos mx \cos nx\,dx = 0$ for $m \neq n$.

26-8. Properties of the Definite Integral. If

$$\int f(x)\,dx = F(x) + C,$$

then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

and

$$\int_b^a f(x)\,dx = F(a) - F(b).$$

Hence,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx; \qquad (26\text{-}27)$$

that is, an interchange of limits is equivalent to a change of sign of the definite integral.

Furthermore, inasmuch as

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

$$\int_b^c f(x)\,dx = F(c) - F(b),$$

and

$$\int_a^c f(x)\,dx = F(c) - F(a),$$

it follows that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{26-28}$$

In the following chapter it will be shown that the definite integral $\int_a^c f(x)\,dx$ represents the area under the curve of $f(x)$ from $x = a$, to $x = c$. In terms of areas, Eq. (26-28) states that the area under the curve of $f(x)$ between $x = a$ and $x = c$ is equal to the sum of the areas under the curve of $f(x)$ from $x = a$ to $x = b$ and from $x = b$ to $x = c$.

CHAPTER 27

INTEGRATION AS SUMMATION

27–1. Area Under a Curve. In Sec. 10–3 we introduced the notion of area under a curve in connection with average values. Here we investigate a means of expressing the area under a curve in terms of the function which describes the curve and of the abscissas of the vertical lines which bound the area on the left and on the right. With reference to

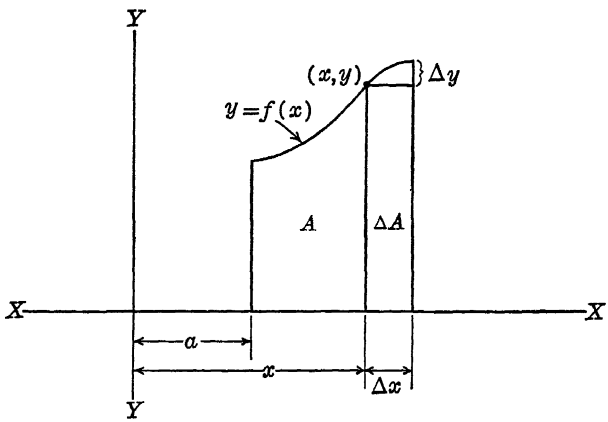


FIG. 27–1. Area under a curve.

Fig. 27–1 let us consider the variable area A which is bounded on the top by $y = f(x)$, on the bottom by the x-axis, on the left by the line $x = a$, and on the right by the line $x = x$. This area is a function of x since it assumes different values for different values of x. Corresponding to an increment Δx in x there is an increment ΔA in A which is such that

$$y\,\Delta x < \Delta A < (y + \Delta y)\,\Delta x.$$

(If $f(x)$ decreases as x increases, the inequality signs are reversed.) Dividing by Δx, we obtain

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

Now allowing Δx to approach the limit zero, we see that

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx} = y = f(x).$$

In terms of differentials,

$$dA = f(x)\,dx, \tag{27-1}$$

from which

$$A = \int dA = \int f(x)\,dx. \tag{27-2}$$

If

$$\int f(x)\,dx = F(x) + C,$$

the area between $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a). \tag{27-3}$$

In a similar manner we may arrive at an expression for the area which is bounded on the left by the y-axis, on the right by the curve $x = g(y)$, on the bottom by the line $y = c$, and on the top by the line $y = d$:

$$A = \int_c^d g(y)\,dy.$$

By Eq. (27-3) the area under the curve of $y = \cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = \sin\theta\Big]_0^{\frac{\pi}{2}} = 1 - 0 = 1; \tag{27-4}$$

and the area under the curve of $y = \sin^2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big]_0^{\frac{\pi}{2}} = \left(\frac{\pi}{4} - 0\right) - (0 - 0) = \frac{\pi}{4}. \tag{27-5}$$

From Eqs. (27-4) and (27-5), respectively, the half-wave average value of $\cos\theta$ is

$$\frac{1}{\frac{\pi}{2}} = \frac{2}{\pi},$$

and the average value of $\sin^2 \theta$ is

$$\frac{\frac{\pi}{4}}{\frac{\pi}{2}} = \frac{1}{2}.$$

These results are the same as those obtained in Chapter 10.

Exercise 27-1

Find the area:

1. Under the curve of $y = \sin \theta$ from $\theta = 0$ to $\theta = \pi$;
2. Under the curve of $y = \cos^2 \theta$ from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.
3. Between the x-axis and the curve of $y = 4x - x^2$, which is shown in Fig. 23-2.
4. Between the line $y = x$ and the curve $y = 4x - x^2$. (Draw the graphs of $y = x$ and $y = 4x - x^2$ on the same set of axes. Determine the points of intersection — the origin is one — by solving the equations simultaneously. Find the area under each curve from the origin on the left to the point of intersection on the right. Subtract these areas.)

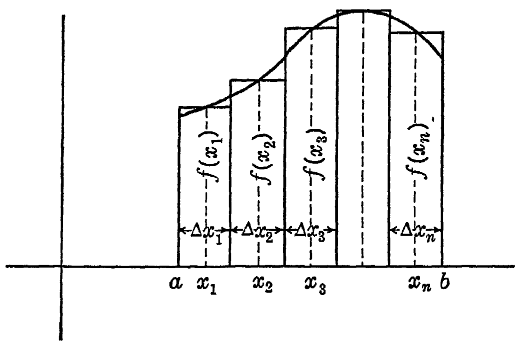


FIG. 27-2. Subdivision of area under curve into elementary rectangular areas.

27-2. The Definite Integral as the Limit of a Sum. For many applications the definite integral is best regarded as the limit of a sum of differentials. Let us consider again a function $f(x)$ which is such that $\int f(x)\, dx = F(x) + C$. In the preceding section we observed that the

area, A, under this curve between $x = a$ and $x = b$ is equal to $F(b) - F(a)$. In Fig. 27-2 this area is subdivided into n elementary rectangular areas each of width Δx_i and of height $f(x_i)$, where i is some number from 1 to n inclusive. The total area of all these rectangles is given by

$$S_n = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

As the width of each rectangle is taken smaller, that is, as the number of rectangles is increased, the total area of the rectangles, S_n, approaches the area A. Thus, we have

$$\int f(x)\, dx = \lim_{n\to\infty} S_n. \tag{27-6}$$

Let us evaluate some areas using the notion of the integral as the limit of a sum. The area under the curve of $\sin\theta$ between $\theta = 0$ and $\theta = \pi$ in Fig. 27-3 is the limit of the sum of the elementary rectangular areas $y\, d\theta$, and this, by Eq. (27-6), is

$$\int_0^\pi y\, d\theta = \int_0^\pi \sin\theta\, d\theta = -\cos\theta\Big]_0^\pi = 2.$$

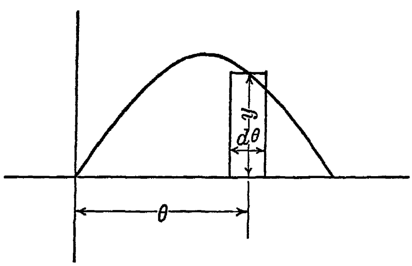


FIG. 27-3. Evaluation of area under loop of sine curve.

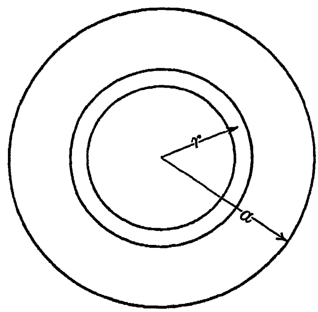


FIG. 27-4. Evaluation of area of circle.

The circular area of Fig. 27-4 is the limit of the sum of the elementary ring areas, $2\pi r\, dr$, and this, by Eq. (27-6), is

$$\int_0^a 2\pi r\, dr = \pi r^2\Big]_0^a = \pi a^2.$$

This same area might have been evaluated by using rectangular ele-

mentary areas, $y\,dx$ as in Fig. 27-5(a), or $x\,dy$ as in Fig. 27-5(b). The equation of the circle is $x^2 + y^2 = a^2$. Thus, the height y of an elementary area in Fig. 27-5(a) is $\sqrt{a^2 - x^2}$, and the length x of an elementary area in Fig. 27-5(b) is $\sqrt{a^2 - y^2}$. We obtain, therefore, for the area of the circle

$$A = 4\int_0^a y\,dx = 4\int_0^a \sqrt{a^2 - x^2}\,dx,$$

or

$$A = 4\int_a^a x\,dy = 4\int_0^a \sqrt{a^2 - y^2}\,dy.$$

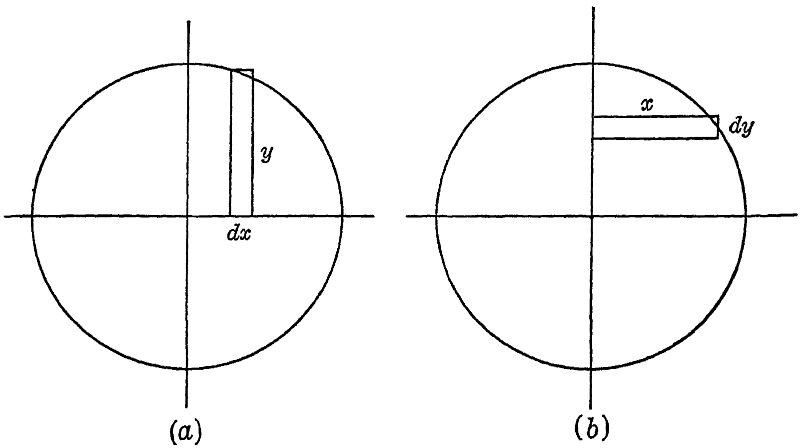


FIG. 27-5. Rectangular elementary areas for evaluation of circular area by integration.

Using the result of Problem B20 of Exercise 26-1, we have

$$\frac{A}{4} = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\Big]_0^a$$

$$= 0 + \frac{\pi a^2}{4},$$

from which

$$A = \pi a^2.$$

Exercise 27-2

By integration find:

1. The area bounded by the line $y = 2x$, the line $x = 5$, and the x-axis.
2. The area of a ring of outer radius r_2 and inner radius r_1.
3. The area under the curve of $y = \epsilon^{-x}$ from $x = 1$ to $x = 10$.

27-3. Volumes by Integration. The notions of Eq. (27-2) may be extended to apply to volumes. Thus, the volume of the right circular cone

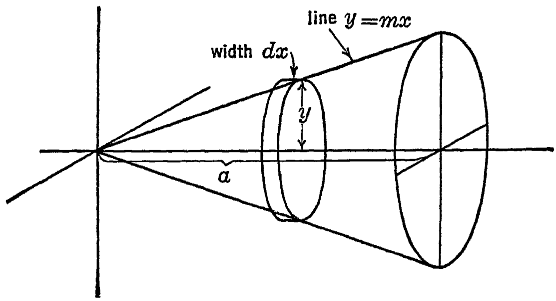


FIG. 27-6. Evaluation of volume of cone.

shown in Fig. 27-6 is obtained as the limit of the sum of the disc elementary volumes, each of width dx and of radius $y = mx$. The elementary volume is given by

$$dV = \pi y^2\,dx = \pi m^2 x^2\,dx,$$

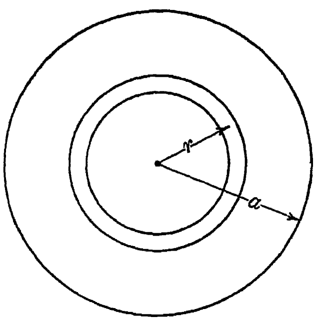


FIG. 27-7. Spherical shell elementary volume used in evaluation of volume of sphere.

and the total volume of the cone is given by

$$V = \int_0^a \pi m^2 x^2\,dx = \frac{\pi m^2 x^3}{3}\Big]_0^a = \frac{\pi m^2 a^3}{3}.$$

The volume of a sphere of radius a may be determined by summing the concentric shell elementary volumes shown in Fig. 27-7. Here each shell is of surface area $4\pi r^2$ and of thickness dr. Hence, it is of volume

$$dV = 4\pi r^2\,dr,$$

and the complete volume of the sphere is

$$V = \int_0^a 4\pi r^2\, dr = \frac{4\pi r^3}{3}\bigg]_0^a = \frac{4\pi a^3}{3}.$$

An alternative procedure is to use a disc element of volume as in the example of the cone. With this arrangement (Fig. 27-8) we have

$$dV = \pi y^2\, dx,$$

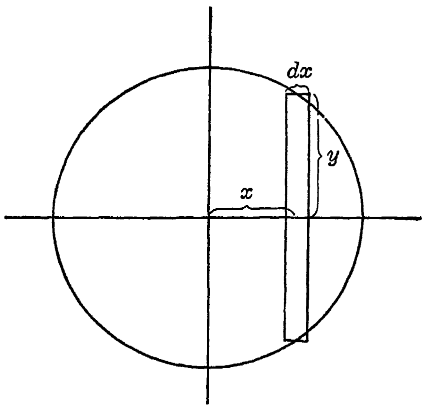


FIG. 27-8. Disc elementary volume used in evaluation of volume of sphere.

where

$$y^2 = a^2 - x^2.$$

Thus,

$$V = 2\int_0^a \pi(a^2 - x^2)\, dx = 2\pi\left[\int_0^a a^2\, dx - \int_0^a x^2\, dx\right]$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4\pi a^3}{3}.$$

Exercise 27-3

By integration:

1. Find the volume of the right circular cone which has a radius of base b and a height h.

2. Find the volume of the frustum of the cone of Problem 1 of altitude a (Fig. 27-9).
3. Find the volume of the oblique circular cylinder which is shown in Fig. 27-10.
4. Find the volume included between concentric spheres of radii r_1 and r_2.

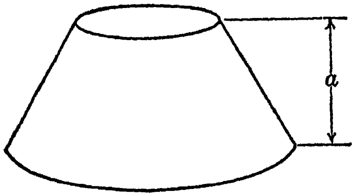


FIG. 27-9. Frustum of cone.

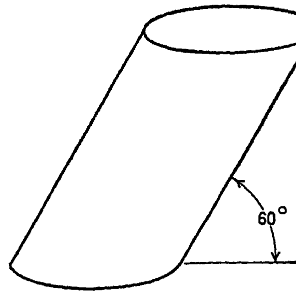


FIG. 27-10. Oblique circular cylinder.

27-4. Miscellaneous Applications of the Definite Integral as the Limit of a Sum. A number of problems in electricity are conveniently treated through the use of the definite integral as the limit of a sum.

By definition, the potential at a particular point in an electrostatic field is the work required to bring a unit positive charge from outside the

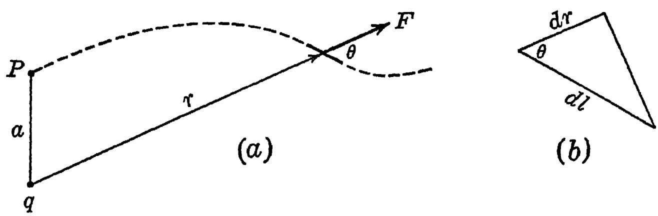


FIG. 27-11. (a) Calculation of the potential at a point. (b) Magnified element of path length.

field (from an infinite distance and along any path) to the point in question. With reference to Fig. 27-11 let us compute the potential at the point P which is at a distance a from the charge q. The dotted line represents the path of the unit positive test charge in approaching the point P. The force of repulsion on the test charge at any point in its path is given by

$$F = \frac{q}{r^2}, \tag{27-7}$$

and the work done in advancing the charge a distance dl towards P is, in accordance with the definition of work,

$$dW = F\,dl \cos\theta. \tag{27-8}$$

In Eqs. (27-7) and (27-8), F is in dynes per esu, q is in esu, dl and dr are in centimeters, and dW is in ergs per esu. The total work in bringing the unit charge from an infinite distance to P, that is, the electric potential at P, is

$$V = \int F \cos\theta\, dl = q \int \frac{\cos\theta\, dl}{r^2}.$$

From Fig. 27-11(b),

$$dr = -dl \cos\theta,$$

the minus sign indicating that dr is negative (decrease in r) when $\cos\theta$ is positive and that dr is positive (increase in r) when $\cos\theta$ is negative. Hence,

$$V = -q \int_{\infty}^{a} \frac{dr}{r^2} = \frac{q}{r}\Big]_{\infty}^{a} = \frac{q}{a}.$$

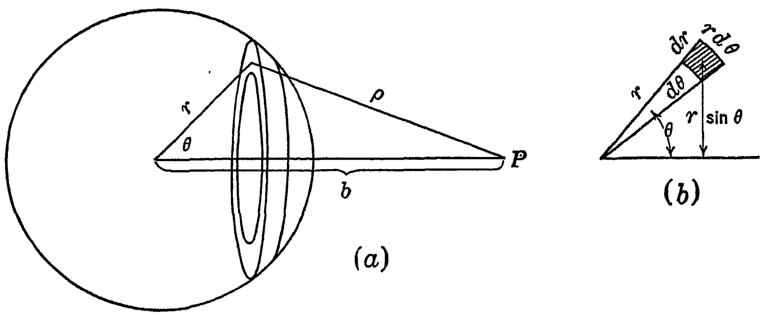


FIG. 27-12. (a) Calculation of potential at a point due to a uniformly charged sphere. (b) Cross section through elementary ring volume.

Potential, being a scalar, is directly additive. The electric potential at a point outside of a uniformly charged sphere may be determined as follows. With reference to Fig. 27-12 we choose for an elementary volume of the sphere the ring of volume

$$d^2v = 2\pi r \sin\theta \cdot dr \cdot r\,d\theta.$$

(We here use v to denote volume, reserving V for potential.) If the

charge density is σ, the charge included in the elementary volume is

$$d^2q = \sigma \cdot 2\pi r^2 \sin\theta \, dr \, d\theta,$$

and the potential at P due to this ring element of charge is

$$\begin{aligned} d^2V &= \frac{d^2q}{\rho} \\ &= \frac{2\pi\sigma \, r^2 \sin\theta \, dr \, d\theta}{\rho}. \end{aligned}$$

Now, by the Law of Cosines,

$$\rho^2 = r^2 + b^2 - 2rb\cos\theta,$$

so that for a fixed radius r

$$2\rho \, d\rho = 2rb \sin\theta \, d\theta.$$

Hence,

$$d^2V = \frac{2\pi\sigma \, r \, dr \, d\rho}{b}.$$

The potential at P due to the spherical shell of radius r is given by

$$\begin{aligned} dV &= \frac{2\pi\sigma \, r \, dr}{b} \int_{b-r}^{b+r} d\rho \\ &= \frac{2\pi\sigma \, r \, dr}{b} [(b + r) - (b - r)] \\ &= \frac{4\pi\sigma \, r^2 \, dr}{b}. \end{aligned}$$

Then, if the sphere is of radius a, the potential at P due to the complete sphere is given by

$$\begin{aligned} V &= \frac{4\pi\sigma}{b} \int_0^a r^2 \, dr \\ &= \frac{4\pi\sigma}{b} \left[\frac{r^3}{3}\right]_0^a \\ &= \frac{4\pi\sigma \, a^3}{3}. \end{aligned}$$

The determination of the potential at a point inside the sphere is performed in a similar manner (Problem 1 of Exercise 27-4).

Magnetic fields and, in turn, inductances, associated with current carrying conductors of various shapes are theoretically determined by a summation process using integrals. The basic relation here is Ampère's Law for the magnetic field at a point due to a current element:

$$dH = \frac{i\,dl \sin \alpha}{r^2}, \tag{27-9}$$

where α and r are as indicated in Fig. 27-13 and where the direction of dH is the direction of the advance of a right-hand screw which turns from i into r. (In Fig. 27-13 this is into the paper.) dH in Eq. (27-9) is in gauss, i is in emu, and dl and r are in centimeters.

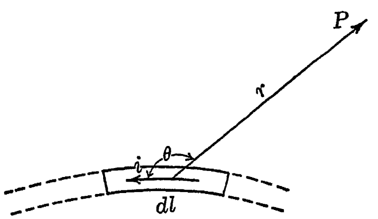


FIG. 27-13. Circuit element for calculation of magnetic field at point P.

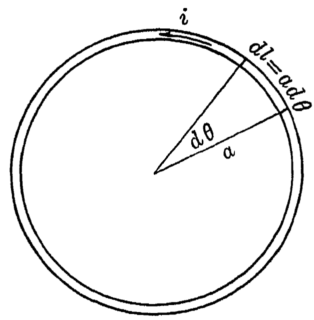


FIG. 27-14. Calculation of magnetic field at the center of a circular conductor by Ampère's Law.

A circular circuit of radius a is shown in Fig. 27-14. Since $dl = a d\theta$, the magnetic field at the center is in magnitude

$$H = \frac{i}{a}\int_0^{2\pi} d\theta = \frac{2\pi i}{a},$$

and in direction out from the paper.

Exercise 27-4

1. Show that the potential of a uniformly charged disc at a point on the axis a distance a from the center is $2\pi\sigma\sqrt{(R^2 + a^2} + a)$, where σ is the charge per unit area and R is the radius of the disc.

2. Show that for the uniformly charged sphere considered in Sec. 27-4 the potential at a point inside the sphere which is a distance b from the center of

the sphere is given by

$$V = 2\pi\sigma a^2.$$

3. Show that the magnetic field at a distance a from an infinitely long straight current is in magnitude

$$H = \frac{2i}{a}.$$

(This is known as the Law of Biot and Savart.)

4. Show that for a circular circuit the magnetic field at a point on the axis of the circle, at a distance x from the center of the circle, is directed along the axis and is of magnitude

$$H = \frac{2\pi i a^2}{(a^2 + x^2)^{\frac{3}{2}}}.$$

5. Show that the magnetic field near the center of a solenoid whose length, l, is very great compared with its radius, is

$$H = \frac{4\pi n i}{l}.$$

6. The potential difference between two points is the work required to move a unit charge from one point to the other against the action of the electrostatic field. Show that the total work in charging a condenser of capacitance C to a potential V is $\frac{CV^2}{2}$. (Capacitance of a condenser is the charge per unit of potential difference, $C = \frac{q}{V}$, where q and V represent corresponding charge and potential difference, respectively, at any particular instant.)

CHAPTER 28

FOURIER SERIES

28–1. The General Problem. A given non-sinusoidal periodic function, such as might represent a voltage or current, may be expanded in a series of sinusoidal terms whose frequencies are integral multiples of the fundamental frequency of the given function. Such a series is called a *Fourier series.*

Let us consider a function $f(x)$ which is recurrent, of period 2π, so that

$$f(x + 2\pi) = f(x). \tag{28–1}$$

Then let us propose to represent this function by the Fourier series:

$$f(x) = a + b_1 \cos x + c_1 \sin x + b_2 \cos 2x + c_2 \sin 2x + \cdots. \tag{28–2}$$

By the infinite series on the right side of Eq. (28–2) we mean the limit which is approached by S_n, the sum of n terms of the series, as n increases indefinitely. Let us presume that over an interval x_1 to x_2 in which we are interested, $f(x)$ is such that

$$\int_{x_1}^{x_2} f(x)\, dx = \int_{x_1}^{x_2} a\, dx + \int_{x_1}^{x_2} b_1 \cos x\, dx + \int_{x_1}^{x_2} c_1 \sin x\, dx + \cdots; \tag{28–3}$$

that is, that $\int_{x_1}^{x_2} f(x)\, dx$ is equal to the limit which is approached by the sum of the terms of the series which appears on the right side of Eq. (28–3). Ordinarily, the functions we encounter in practice do possess this property. It is by means of Eq. (28–3) that we are able to evaluate the coefficients $a, b_1, c_1 \cdots$, of Eq. (28–2) and, hence, to develop a desired Fourier expansion.

In the evaluation of coefficients we shall require the following definite integrals. (See Problems 6 through 12 of Exercise 26–4):

$$\int_{k}^{k+2\pi} \frac{dx}{2} = \pi; \tag{28–4}$$

$$\int_k^{k+2\pi} \sin nx\,dx = 0; \tag{28-5}$$

$$\int_k^{k+2\pi} \cos nx\,dx = 0; \tag{28-6}$$

$$\int_k^{k+2\pi} \sin^2 nx\,dx = \pi; \tag{28-7}$$

$$\int_k^{k+2\pi} \cos^2 nx\,dx = \pi; \tag{28-8}$$

$$\int_k^{k+2\pi} \sin mx \cos nx\,dx = 0; \tag{28-9}$$

$$\int_k^{k+2\pi} \sin mx \sin nx\,dx = 0 \text{ for } m \neq n; \tag{28-10}$$

$$\int_k^{k+2\pi} \cos mx \cos nx\,dx = 0 \text{ for } m \neq n. \tag{28-11}$$

On integrating both sides of Eq. (28-2) from k to $k + 2\pi$, we obtain

$$\int_k^{k+2\pi} f(x)\,dx = a\int_k^{k+2\pi} dx = 2\pi a. \tag{28-12}$$

All the other integrals on the right side of the equation vanish because they are of the type of either Eq. (28-5) or Eq. (28-6). On multiplying both sides of Eq. (28-3) by $\cos nx$ and integrating from k to $k + 2\pi$, we obtain

$$\int_k^{k+2\pi} f(x)\cos nx\,dx = b_n\int_k^{k+2\pi} \cos^2 nx\,dx = \pi b_n. \tag{28-13}$$

All the other integrals on the right side of the equation vanish because they are of the type of either Eq. (28-9) or Eq. (28-11). Similarly, on multiplying both sides of Eq. (28-3) by $\sin nx$ and integrating from k to $k + 2\pi$, we obtain

$$\int_k^{k+2\pi} f(x)\sin nx\,dx = c_n\int_k^{k+2\pi} \sin^2 nx\,dx = \pi c_n. \tag{28-14}$$

All the other integrals on the right side of the equation vanish because they are of the type of either Eq. (28-9) or Eq. (28-10).

$$\int_k^{k+2\pi} \sin nx\,dx = 0; \tag{28-5}$$

$$\int_k^{k+2\pi} \cos nx\,dx = 0; \tag{28-6}$$

$$\int_k^{k+2\pi} \sin^2 nx\,dx = \pi; \tag{28-7}$$

$$\int_k^{k+2\pi} \cos^2 nx\,dx = \pi; \tag{28-8}$$

$$\int_k^{k+2\pi} \sin mx \cos nx\,dx = 0; \tag{28-9}$$

$$\int_k^{k+2\pi} \sin mx \sin nx\,dx = 0 \text{ for } m \neq n; \tag{28-10}$$

$$\int_k^{k+2\pi} \cos mx \cos nx\,dx = 0 \text{ for } m \neq n. \tag{28-11}$$

On integrating both sides of Eq. (28–2) from k to $k + 2\pi$, we obtain

$$\int_k^{k+2\pi} f(x)\,dx = a\int_k^{k+2\pi} dx = 2\pi a. \tag{28-12}$$

All the other integrals on the right side of the equation vanish because they are of the type of either Eq. (28–5) or Eq. (28–6). On multiplying both sides of Eq. (28–3) by $\cos nx$ and integrating from k to $k + 2\pi$, we obtain

$$\int_k^{k+2\pi} f(x)\cos nx\,dx = b_n\int_k^{k+2\pi} \cos^2 nx\,dx = \pi b_n. \tag{28-13}$$

All the other integrals on the right side of the equation vanish because they are of the type of either Eq. (28–9) or Eq. (28–11). Similarly, on multiplying both sides of Eq. (28–3) by $\sin nx$ and integrating from k to $k + 2\pi$, we obtain

$$\int_k^{k+2\pi} f(x)\sin nx\,dx = c_n\int_k^{k+2\pi} \sin^2 nx\,dx = \pi c_n. \tag{28-14}$$

All the other integrals on the right side of the equation vanish because they are of the type of either Eq. (28–9) or Eq. (28–10).

Eqs. (28–12), (28–13), and (28–14) provide a means of evaluating the coefficients of Eq. (28–2). Thus, from Eq. (28–12),

$$a = \frac{1}{2\pi}\int_k^{k+2\pi} f(x)\,dx; \tag{28–15}$$

from Eq. (28–13),

$$b_n = \frac{1}{\pi}\int_k^{k+2\pi} f(x)\cos nx\,dx \text{ for } n = 1, 2, 3 \cdots ; \tag{28–16}$$

and, from Eq. (28–14),

$$c_n = \frac{1}{\pi}\int^{k+2\pi} f(x)\sin nx\,dx \text{ for } n = 1, 2, 3 \cdots . \tag{28–17}$$

Eq. (28–2) with the constants determined by Eqs. (28–15), (28–16), and (28–17), is known as a *formal development* of the function $f(x)$, since the series on the right of Eq. (28–2) exists in form regardless of considerations of the validity of our operations.

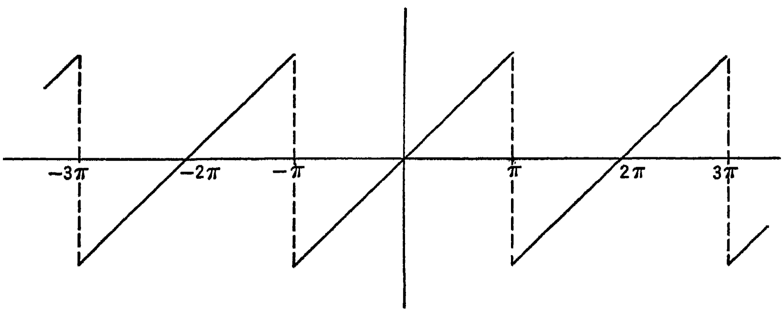


FIG. 28–1. Saw tooth function.

28–2. Specific Cases. We now expand in Fourier series some simple functions of the type represented by Eq. (28–1). Let us first consider the saw tooth function $f(x)$ shown in Fig. 28–1. This function is of the type $f(x + 2\pi) = f(x)$. If we can develop a Fourier series which will represent this function for all values of x between $-\pi$ and π, we shall then have an expansion which represents the function for any value of x. Inasmuch as for any angle x and for any integer n,

$$\sin n(x + 2\pi) = \sin nx$$

and

$$\cos n(x + 2\pi) = \cos nx$$

it follows that the value of the Fourier expansion is the same for $x + 2\pi$ as for x; and this is consistent with the periodic nature of the function $f(x)$ as shown in the graph or as expressed by Eq. (28-1).

For $-\pi < x < \pi$ we have $f(x) = x$. Then, by Eqs. (28-15), (28-16), and (28-17),

$$a = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{x^2}{4\pi}\Big]_{-\pi}^{\pi} = 0;$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi n^2}\Big[\cos nx + nx\sin nx\Big]_{-\pi}^{\pi} = 0;$$

$$c_n = \frac{1}{\pi}\int_{\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi n^2}\Big[\sin nx - nx\cos nx\Big]_{-\pi}^{\pi} = -\frac{2\cos n\pi}{n}.$$

The equation for c_n yields

$$c_1 = \frac{2}{1},\ c_2 = -\frac{2}{2},\ c_3 = \frac{2}{3},\ \cdots.$$

Hence, for the formal development, we have

$$x = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \cdots).$$

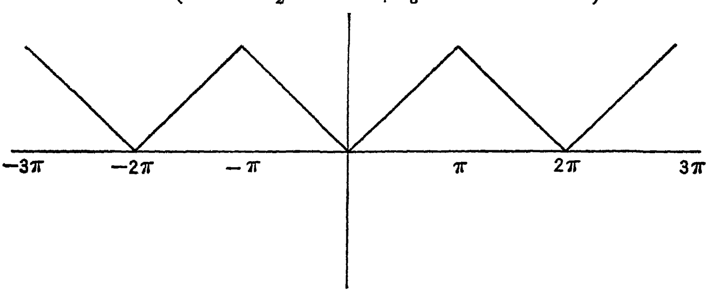


FIG. 28-2. Triangular wave function.

Let us consider next the function shown in Fig. 28-2. Here for $-\pi < x < 0, f(x) = -x$; and for $0 < x < \pi, f(x) = x$. Thus,

$$\begin{aligned} a &= \frac{1}{2\pi}\int_{-\pi}^{0}(-x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx \\ &= \frac{1}{2\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi} \\ &= \frac{\pi^2}{4\pi} + \frac{\pi^2}{4\pi} = \frac{\pi}{2}; \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} (-x) \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x \cos nx\,dx \\
&= -\frac{1}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi} \\
&= -\frac{1}{\pi n^2} + \frac{\cos n\pi}{\pi n^2} + \frac{\cos n\pi}{\pi n^2} - \frac{1}{\pi n^2} = \frac{2(\cos n\pi - 1)}{\pi n^2}; \\
c_n &= \frac{1}{\pi}\int_{-\pi}^{0} (-x) \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx\,dx \\
&= -\frac{1}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi} \\
&= \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} = 0.
\end{aligned}
$$

The equation for b_n yields

$$b_1 = -\frac{4}{\pi},\ b_2 = 0,\ b_3 = -\frac{4}{3^2\pi},\ b_4 = 0,\ b_5 = -\frac{4}{5^2\pi},\ b_6 = 0,\ \cdots.$$

Hence, the formal development is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right).$$

Exercise 28–1

Formally develop each of the following functions in a Fourier series:

1. $f(x) = x^2$ for $-\pi < x < \pi$.
2. $f(x) = -x^2$ for $-\pi < x < 0$; $f(x) = x^2$ for $0 < x < \pi$.
3. $f(x) = \sin x$ for $0 < x < \pi$; $f(x) = 0$ for $\pi < x < 2\pi$.

28–3. Even and Odd Functions. A function $f(x)$ is said to be an *even function* of x if $f(-x) = f(x)$, or an *odd function* of x if $f(-x) = -f(x)$. The function shown in Fig. 28–2 is an even function of x, and the function shown in Fig. 28–1 is an odd function of x. In the Fourier expansion for the even function of Fig. 28–2 all the coefficients of the sine terms were found to be zero, and in the Fourier expansion for the odd function of Fig. 28–1 all the coefficients of the cosine terms were found to be zero. These relations, vanishing of the sine terms in one case and vanishing of the cosine terms in another case, are characteristic of even and odd functions, respectively.

Let us consider an even function $f(x)$ as shown in Fig. 28-3(a). The equation for the evaluation of the coefficient of the term in $\sin nx$ in the Fourier expansion of the function $f(x)$ is

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

The function $\sin nx$ (for $n = 4$) is shown in Fig. 28-3(b), and the product function $f(x) \sin nx$ is shown in Fig. 28-3(c). It will be noted that the area under the graph of the product function, $f(x) \sin nx$, is equal to

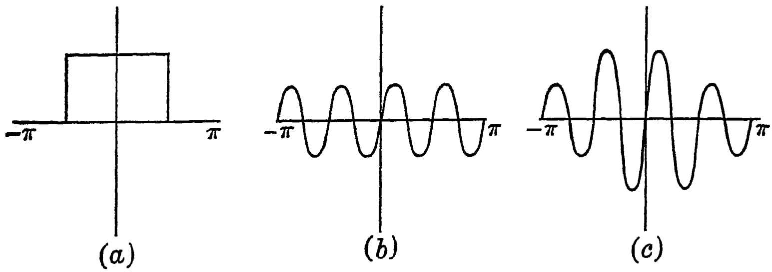


FIG. 28-3. (a) Even function $f(x)$; (b) $\sin nx$; (c) product function $f(x) \sin nx$.

zero since as much of the area lies above the x-axis as below the x-axis. In other words, the integral $\int_{-\pi}^{\pi} f(x) \sin nx \, dx$ is zero. Thus, we see that the Fourier development from $-\pi$ to π for any even function of x contains no sine terms. In a similar manner it may be shown that the integrals $\int_{-\pi}^{\pi} f(x) \, dx$ and $\int_{-\pi}^{\pi} f(x) \cos nx \, dx$ vanish when $f(x)$ is an odd function of x and, hence, that the Fourier development from $-\pi$ to π for any odd function of x contains no constant term and no cosine terms.

Exercise 28-2

A. Classify the following functions as even, odd, or neither:

1. $\sin x$.
2. $\cos 3x$.
3. x^2.
4. x^3.
5. $-7x^2$.
6. $x^2 + 2$.

B. Show that:

1. The Fourier development from $-\pi$ to π for any odd function of x contains no constant term.
2. The Fourier development from $-\pi$ to π for any odd function of x contains no cosine terms.

C. Formally develop each of the following functions in a Fourier series, first determining the non-zero terms by inspection so that only these terms need to be evaluated:

1. $f(x) = 0$ for $-\pi < x < 0$; $f(x) = 1$ for $0 < x < \pi$.
2. $f(x) = 0$ for $-\pi < x - \dfrac{\pi}{2}$ and $\dfrac{\pi}{2} < x < \pi$; $f(x) = 1$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

28–4. Approximations. An adequate approximation to a given function is often obtained by using simply the first few terms of the Fourier expansion for the function. Portions of the graphs of the functions of Figs. 28–1 and 28–2 are shown in Figs. 28–4(a) and 28–4(b), respectively; and beneath each are shown the successive approximations obtained by using first one, then two, and then three terms of the corresponding Fourier series. Thus, the top figure in each column of Fig. 28–4 shows the given function. The second figure represents the first approximation, which is formed by the first term of the series. The third figure represents the second approximation, which is formed by adding the second term of the series to the first approximation. The fourth figure represents the third approximation, which is formed by adding the third term of the series to the second approximation.

Exercise 28–3

1. Extend the successions of approximations in Figs. 28–4(a) and (b) by sketching the graphs obtained on adding one more term of the appropriate Fourier series in each case.

2. Construct graphs corresponding to those of Fig. 28–4 for each of the functions of Part C of Exercise 28–2.

Exercise 28–4

When a non-sinusoidal wave of current passes through the current coil of a wattmeter, the amplitudes of the component sine and cosine waves present in the current can be determined one by one by applying to the voltage coil of the wattmeter pure sine waves of frequencies equal to the frequencies present in the current wave. If nf is the frequency of one of the current components,

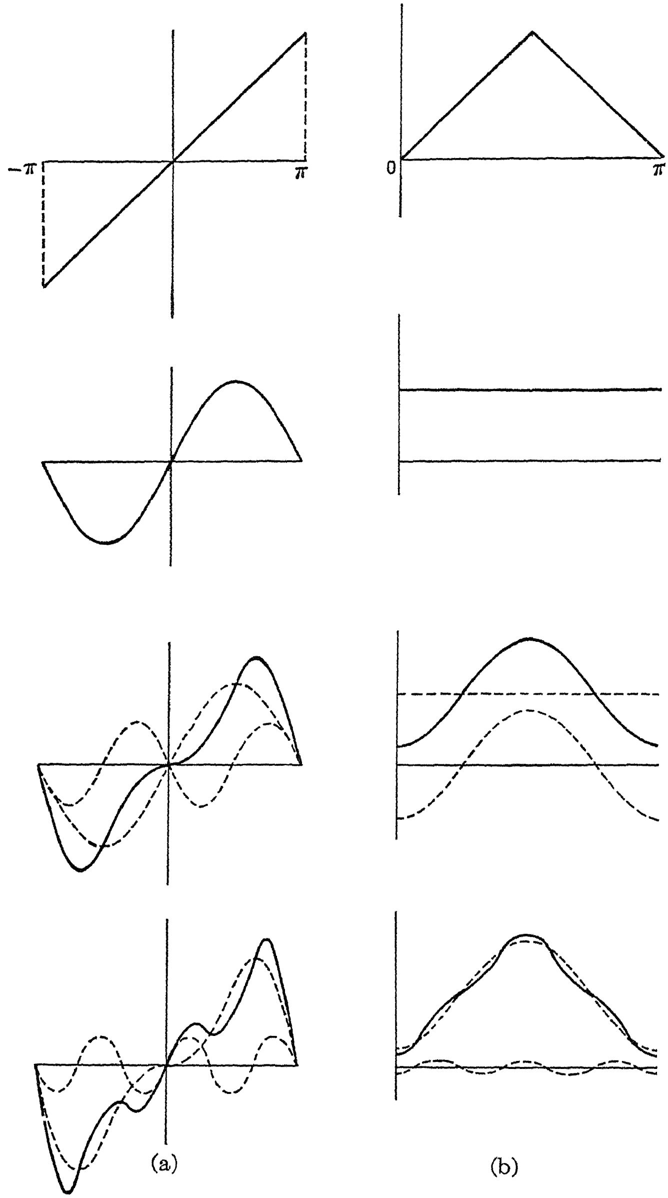


FIG. 28-4. Successive approximations.

then when a sine wave voltage of frequency nf is applied to the voltage coil of the wattmeter, a response of the wattmeter is observed. The reading of the wattmeter is proportional to the magnitude of the current component of frequency nf. Specifically, if

$$i = I_0 + A_1 \sin 2\pi ft + B_1 \cos 2\pi ft + A_2 \sin 2\pi(2f)t + B_2 \cos 2\pi(2f)t + A_3 \sin 2\pi(3f)t + \cdots$$

and, if

$$e = E_n \sin 2\pi(nf)t,$$

then

$$\begin{aligned} P &= \int_0^\tau ie\, dt \\ &= \int_0^\tau I_0E_n \sin 2\pi(nf)t\, dt + \int_0^\tau AE_n \sin 2\pi ft \sin 2\pi(nf)t\, dt + \cdots, \end{aligned}$$

where τ represents the time of one period. If we put $\theta = 2\pi(mf)t$, so that $d\theta = 2\pi(mf)dt$, then we see that an integral of the form $\int_0^\tau \sin 2\pi(mf)t \sin 2\pi(nf)t\, dt$ is equivalent to an integral

$$\frac{1}{2\pi \cdot mf}\int_0^\pi \sin m\theta \sin n\theta\, d\theta;$$

and the latter integral is equal to zero. Likewise we note that every integral in the expression for P vanishes with the exception of one. The wattmeter reading corresponding to an applied voltage $e = E_n \sin 2\pi(nf)t$ is simply

$$P_n = \int_0^\tau A_nE_n \sin^2 2\pi(nf)t\, dt.$$

In an experimental harmonic analysis, using a wattmeter in conjunction with a variable frequency sine wave voltage source as described above, the following observations were recorded:

f in cycles per second	0(d-c)	60	120	180	240	300
E in volts	100	100	100	100	100	100
I in amperes	1.8	2.4	2.5	1.2	1.2	1.2
P in watts	182	224	96	4	11	2

From these data compute the harmonic amplitudes of the current. (Bear in mind that the ammeter and voltmeter readings are effective values.) Plot successive approximations to the wave form of the current in the manner of Fig. 28-4.

CHAPTER 29

SIMULTANEOUS LINEAR EQUATIONS

29–1. Systems of Three Linear Equations in Three Variables. In Chapter 8 we introduced determinants in the solution of simultaneous linear equations in two variables. Here we shall extend the method to the solution of systems of n simultaneous linear equations in n variables.

Let us first solve a system of three linear equations in three variables by straightforward methods. For a specific problem we consider the system of equations

$$3x - y + 2z = 9; \tag{29–1}$$

$$2x + y - z = 7; \tag{29–2}$$

$$x + 2y - 3z = 4. \tag{29–3}$$

Solving each equation in turn for x, we obtain:

$$x = \frac{y}{3} - \frac{2z}{3} + 3; \tag{29–4}$$

$$x = -\frac{y}{2} + \frac{z}{2} + \frac{7}{2}; \tag{29–5}$$

$$x = -2y + 3z + 4. \tag{29–6}$$

The three original equations, therefore, impose the following restrictions on y and z:

$$\frac{y}{3} - \frac{2z}{3} + 3 = -\frac{y}{2} + \frac{z}{2} + \frac{7}{2} \tag{29–7}$$

and

$$\frac{y}{3} - \frac{2z}{3} + 3 = -2y + 3z + 4. \tag{29–8}$$

The above relations constitute a system of two simultaneous equations in the variables y and z the solution of which follows the methods of Chapter 8. The solution is $y = 2$, $z = 1$. Substitution of these values of y and z into any one of the original equations yields $x = 3$.

Eq. (29–7) above was obtained from Eqs. (29–4) and (29–5). There is a third equation in y and z obtainable from the equations (29–1) through (29–3), but the third equation in y and z is not independent of the ones which were chosen. The third equation,

$$-\frac{y}{2}+\frac{z}{2}+\frac{7}{2} = -2y + 3z + 4, \tag{29-9}$$

follows on equating the right sides of Eqs. (29–5) and (29–6). However, it also follows on subtracting Eq. (29–8) from Eq. (29–7). In other words, Eq. (29–9) introduces nothing beyond what is already expressed in Eqs. (29–7) and (29–8). Of the three equations in y and z which arise from Eqs. (29–4) through (29–6) the choice of any particular pair for the determination of y and z is arbitrary, and the third equation is redundant. In general, n linear equations in n quantities are necessary and sufficient uniquely to determine each of the n quantities.

Exercise 29–1

A. **1.** Solve Eqs. (29–7) and (29–9) for y and z.

2. Solve Eqs. (29–8) and (29–9) for y and z.

B. Solve the following systems of equations by the method of Sec. 29–1:

1. $x + 2y + z = 7;$
$x + y - z = 2;$
$3x - y + 2z = 12.$

2. $2x + y - 2z = 3;$
$3x - 2y + 3z = 9;$
$z = x + y.$

29–2. Solution of System of Three Linear Equations in Three Variables by the Method of Determinants. For a system of equations

$$A_1x + B_1y + C_1z = D_1, \tag{29-10}$$

$$A_2x + B_2y + C_2z = D_2, \tag{29-11}$$

$$A_3x + B_3y + C_3z = D_3, \tag{29-12}$$

an extension of the method of determinants which was introduced in Sec. 8–2 leads to a convenient general form for expressing the solution:

$$x = \frac{\begin{vmatrix} D_1 & B_1 & C_1 \\ D_2 & B_2 & C_2 \\ D_3 & B_3 & C_3 \end{vmatrix}}{\Delta}; \tag{29-13}$$

$$y = \frac{\begin{vmatrix} A_1 & D_1 & C_1 \\ A_2 & D_2 & C_2 \\ A_3 & D_3 & C_3 \end{vmatrix}}{\Delta}; \tag{29-14}$$

$$z = \frac{\begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix}}{\Delta}; \tag{29-15}$$

where Δ represents the determinant

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

The determinant, Δ, is formed from the coefficients of x, y, and z as they are arranged in Eqs. (29–10), (29–11), and (29–12), respectively. The numerator in the expression for x in Eq. (29–13) differs from Δ in form only in that the constants D_1, D_2, and D_3 replace the coefficients of x: C_1, C_2, and C_3. In the numerator of the expression for y in Eq. (29–14) the D's replace the coefficients of y: B_1, B_2, and B_3; and in the numerator of the expression for z in Eq. (29–15) the D's replace the coefficients of z: C_1, C_2, and C_3. The determinants used here are called *third order determinants* since each has three rows and three columns. The determinants employed in Sec. 8–3 are called second order determinants.

The evaluation of each determinant in Eqs. (29–13), (29–14), and (29–15) is as illustrated for Δ by the diagram of Fig. 29–1. The array of Fig. 29–1 is formed by rewriting the first and second columns to the right of the determinant. Products of three terms each are then formed: (a) from the sets of elements joined by solid lines, and (b) from the sets of elements joined by dotted lines. To each of the latter group of products a minus sign is attached. The sum of these individual products is the value of the determinant. Δ is, thus, equal to

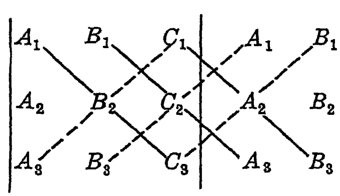


FIG. 29–1. Associated product terms in the evaluation of a determinant of the third order.

$$A_1B_2C_3 + B_1C_2A_3 + C_1A_2B_3 - A_3B_2C_1 - B_3C_2A_1 - C_3A_2B_1.$$

Example. Solve the system of equations:

$$\begin{aligned} 2x + 5y - 3z &= -9; \\ 6x + 3y + 2z &= 7; \\ 3x - 2y + 4z &= 13. \end{aligned}$$

By Eqs. (29–13), (29–14), and (29–15)

$$x = \frac{\begin{vmatrix} -9 & 5 & -3 \\ 7 & 3 & 2 \\ 13 & -2 & 4 \end{vmatrix}}{\Delta};$$

$$y = \frac{\begin{vmatrix} 2 & -9 & -3 \\ 6 & 7 & 2 \\ 3 & 13 & 4 \end{vmatrix}}{\Delta};$$

and

$$z = \frac{\begin{vmatrix} 2 & 5 & -9 \\ 6 & 3 & 7 \\ 3 & -2 & 13 \end{vmatrix}}{\Delta};$$

where

$$\Delta = \begin{vmatrix} 2 & 5 & -3 \\ 6 & 3 & 2 \\ 3 & -2 & 4 \end{vmatrix}.$$

Writing the first two columns of Δ to the right of Δ, we have

$$\begin{vmatrix} 2 & 5 & -3 \\ 6 & 3 & 2 \\ 3 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & 5 \\ 6 & 3 \\ 3 & -2. \end{matrix}$$

Then forming products as in Fig. 29–1, we obtain

$$2 \cdot 3 \cdot 4 + 5 \cdot 2 \cdot 3 + (-3) \cdot 6 \cdot (-2) - 3 \cdot 3 \cdot (-3) - (-2) \cdot 2 \cdot 2 - 4 \cdot 6 \cdot 5$$
$$= 24 + 30 + 36 + 27 + 8 - 120 = 5.$$

Following the same procedure with the numerator determinant of the expression for x, we have

$$\begin{vmatrix} -9 & 5 & -3 \\ 7 & 3 & 2 \\ 13 & -2 & 4 \end{vmatrix} \begin{matrix} -9 & 5 \\ 7 & 3 \\ 13 & -2, \end{matrix}$$

from which we obtain for the value of the corresponding determinant:

$$(-9) \cdot 3 \cdot 4 + 5 \cdot 2 \cdot 13 + (-3) \cdot 7 \cdot (-2) - 13 \cdot 3 \cdot (-3) - (-2) \cdot 2 \cdot (-9)$$
$$- 4 \cdot 7 \cdot 5 = -108 + 130 + 42 + 117 - 36 - 140 = 5.$$

In the same manner we find -5 for the value of the numerator determinant in the expression for y, and 10 for the value of the numerator determinant in

the expression for z. Thus,

$$x = \frac{5}{5} = 1;$$

$$y = \frac{-5}{5} = -1;$$

and

$$z = \frac{10}{5} = 2.$$

The result may be verified by substituting $x = 1$, $y = -1$, and $z = 2$ into each of the three original equations. Substitution into $2x + 5y - 3z = -9$ yields

$$\begin{aligned} 2 \cdot (1) + 5 \cdot (-1) - 3 \cdot (2) &= -9; \\ 2 - 5 - 6 &= -9; \\ -9 &= -9. \end{aligned}$$

Substitution into $6x + 3y + 2z = 7$ yields

$$\begin{aligned} 6 \cdot (1) + 3 \cdot (-1) + 2 \cdot (2) &= 7; \\ 6 - 3 + 4 &= 7; \\ 7 &= 7. \end{aligned}$$

Substitution into $3x - 2y + 4z = 13$ yields

$$\begin{aligned} 3 \cdot (1) - 2 \cdot (-1) + 4 \cdot (2) &= 13; \\ 3 + 2 + 8 &= 13; \\ 13 &= 13. \end{aligned}$$

$x = 1$, $y = -1$, $z = 2$ is found to satisfy all three given equations. It is, therefore, the correct solution of the given system of equations.

Exercise 29-2

Solve each of the systems of Exercise 29-1 by the method of Sec. 30-2.

29-3. General Method for Evaluation of Determinants. For systems of n linear equations in n variables the formal solution by means of determinants follows the pattern given for the solution of two equations in two variables in Eqs. (8-5) and (8-6), and the pattern given for the solution of three variables in Eqs. (29-13), (29-14), and (29-15). However, the schemes which were used in Secs. 8-3 and 29-2 for the evaluation of determinants of the second order and third order, respectively, do not admit of generalization to the evaluation of a determinant of the nth order. A method of evaluation which applies in any case, including that of $n = 2$ and $n = 3$, involves expansion of the determinant by *minors*.

The minor of any given element in a determinant is defined as that sub-determinant which is made up from the original determinant by deleting the row and column in which the particular element occurs. In the determinant

$$D = \begin{vmatrix} 5 & 1 & 0 & 1 \\ 7 & 6 & 3 & -2 \\ -1 & 2 & 4 & 2 \\ -4 & 0 & 0 & 1 \end{vmatrix},$$

the minor of the element 3 is the sub-determinant formed by deleting the elements of the second row and third column in which the 3 appears:

$$\begin{vmatrix} 5 & 1 & 1 \\ -1 & 2 & 2 \\ -4 & 0 & 1 \end{vmatrix}.$$

The value of any given determinant is obtained through an expansion by minors in terms of the elements of any one row or column. In terms of the elements of the second row

$$D = \begin{vmatrix} 5 & 1 & 0 & 1 \\ 7 & 6 & 3 & -2 \\ -1 & 2 & 4 & 2 \\ -4 & 0 & 0 & 1 \end{vmatrix} = -(7) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} + (6) \cdot \begin{vmatrix} 5 & 0 & 1 \\ -1 & 4 & 2 \\ -4 & 0 & 1 \end{vmatrix}$$

$$-(3) \cdot \begin{vmatrix} 5 & 1 & 1 \\ -1 & 2 & 2 \\ -4 & 0 & 1 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 5 & 1 & 0 \\ -1 & 2 & 4 \\ -4 & 0 & 0 \end{vmatrix}. \qquad (29\text{–}16)$$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

FIG. 29–2. Checkerboard pattern for signs of terms in expansion of determinant by minors. A plus sign always occupies the upper left corner.

The sign of each term in Eq. (29–16) is chosen in accordance with the checkerboard pattern of Fig. 29–2. The signs prefixing the elements 7, 6, 3, and (-2) in the expansion of Eq. (29–16) are the signs which in Fig. 29–2 occupy positions corresponding to each of these elements in the original determinant D. It is adequate to ascertain the sign of only the first term in any such expansion since the signs of the remaining terms alternate in sequence. In a checkerboard sign pattern for any determinant a plus sign always occupies the upper left corner. The expansion of Eq. (29–16) yields for the value

of D

$$-7 \cdot (4) + 6 \cdot (4) - 3 \cdot (11) - 2 \cdot (-16) = -5.$$

Expansion of the same given determinant D in terms of the elements of its first column yields

$$(5) \cdot \begin{vmatrix} 6 & 3 & -2 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} - (7) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 6 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$-(-4) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 6 & 3 & -2 \\ 2 & 4 & 2 \end{vmatrix}$$

$$= 5 \cdot (18) - 7 \cdot (4) - 1 \cdot (3) + 4 \cdot (-16) = -5.$$

Exercise 29-3

1. Give the minors of each element of the first column in the following determinants:

$$\begin{vmatrix} 2 & 6 & -3 & 6 \\ 1 & 4 & 7 & -2 \\ 2 & 1 & 3 & 6 \\ -1 & 0 & 5 & 3 \end{vmatrix}; \quad \begin{vmatrix} 4 & 7 & -2 \\ 1 & 3 & 6 \\ 0 & 5 & 3 \end{vmatrix}; \quad \begin{vmatrix} 3 & 6 \\ 5 & 3 \end{vmatrix}.$$

2. Evaluate first by the method of Sec. 29-2, and then by expansion with minors:

$$\begin{vmatrix} 0 & 1 & 2 \\ -2 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}.$$

3. a. Evaluate $\begin{vmatrix} 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & 0 & 7 \end{vmatrix}$ and $\begin{vmatrix} 2 & 1 & 6 \\ 3 & 2 & 4 \\ 0 & 0 & 0 \end{vmatrix}$.

(Note that the first determinant has one column of all zeros, and the second determinant has one row of all zeros.)

b. Evaluate $\begin{vmatrix} 3 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 2 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 1 \\ 4 & 0 & 5 \\ 4 & 0 & 5 \end{vmatrix}$.

(Note that the first determinant has two columns which are identical, and the second determinant has two rows which are identical.)

c. Evaluate and compare $\begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$ and $\begin{vmatrix} 0 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

The elements of the second determinant are the same as those of the firs with their positions changed. Specify the manner of the rearrangement whic yields the second determinant from the first.

d. Evaluate and compare $\begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$, $\begin{vmatrix} 0 & 1 & (3\cdot2) \\ 2 & 1 & (3\cdot3) \\ 4 & 2 & (3\cdot1) \end{vmatrix}$, and

$$\begin{vmatrix} 0 & 1 & 2 \\ (-4\cdot2) & (-4\cdot1) & (-4\cdot3) \\ 4 & 2 & 1 \end{vmatrix}.$$

Specify the manner in which the elements of the second and third determ nants differ from the first.

e. Evaluate and compare $\begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$, $\begin{vmatrix} 0 & (1+2) & 2 \\ 2 & (1+3) & 3 \\ 4 & (2+1) & 1 \end{vmatrix}$, and

$$\begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ (4+0) & (2+1) & (1+2) \end{vmatrix}.$$

Specify the manner in which the elements of the second and third determ nants differ from the first.

f. Evaluate and compare $\begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$, $\begin{vmatrix} 0 & (1+3\cdot2) & 2 \\ 2 & (1+3\cdot3) & 3 \\ 4 & (2+3\cdot1) & 1 \end{vmatrix}$,

$$\begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ (4-2\cdot2) & (2-2\cdot1) & (1-2\cdot3) \end{vmatrix}.$$

Specify the manner in which the elements of the second and third determ nants differ from the first.

4. Formulate (do not prove) probable theorems on determinants which a exemplified by the results of Problem 3 above.

5. Solve each of the systems of Exercise 29-1 B by the use of determinan first simplifying each determinant where feasible by application of the the rems of Problem 4 above, and then expanding the resulting determinar by minors.

Exercise 29-4

1. For the circuit shown in Fig. 29-3, Kirchhoff's Laws give rise to the following equations:

$$-0.2I_1 - 10I_1 + 0.2I_2 = 120,$$
$$-0.2I_2 + 20I_3 + 0.2I_3 = 120,$$
$$-I_2 = I_1 + I_3.$$

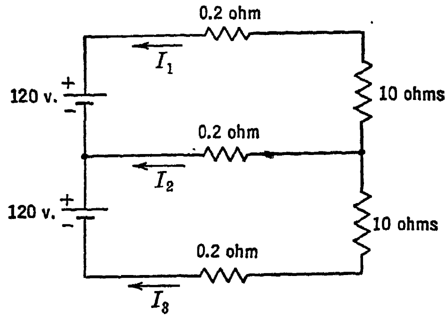


FIG. 29-3. Three-wire distribution system.

Solve these equations for the currents in the various parts of the circuit. (A positive value of a current indicates that the direction of the current is correct as assumed in Fig. 29-3. A negative value of the current indicates that the current is in the opposite direction to that assumed in Fig. 29-3.)

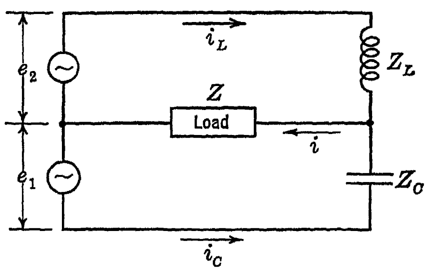


FIG. 29-4. Constant current system.

2. Electrical energy is commonly available from power companies at a relatively fixed potential. In a 118-volt system the terminal potential is maintained at approximately 118 volts whether the load requires 10 watts or 10 horsepower. Certain engineering applications, however, demand not constant potential but constant current for their operation. The fundamental circuit of a system to provide constant current through any particular device from a constant voltage a-c source is shown in Fig. 29-4. The electrical relations

in the circuit are expressed by

$$i = i_L + i_C; \tag{29-17}$$

$$e_1 = i_L Z_L + iZ; \tag{29-18}$$

and

$$e_2 = i_C Z_C + iZ; \tag{29-19}$$

where Z_L, Z_C, and Z are the impedances of the inductor, capacitor, and load, respectively, and i_L, i_C, and i are the corresponding currents. From Eqs. (29-17), (29-18), and (29-19), it follows that

$$i = \frac{e_1 Z_C + e_2 Z_L}{Z_L Z_C + (Z_L + Z_C)Z}. \tag{29-20}$$

If the inductor and capacitor are chosen of appropriate dimensions, so that $Z_L = Z_C$, then the load current is

$$i = \frac{e_1 - e_2}{Z_C}, \tag{29-21}$$

so that the current through the load is independent of the nature of the load and is controlled entirely by e_1, e_2, and Z_C. Eq. (29-21) shows, in other words, that once e_1, e_2, and Z_C are set, the current through the load is constant regardless of whether the load requires 10 watts or 10 horsepower.

Solve Eqs. (29-17), (29-18), and (29-19) for i, and show that Eq. (29-20) reduces to Eq. (29-21) under the condition that $Z_L = -Z_C$.

3.

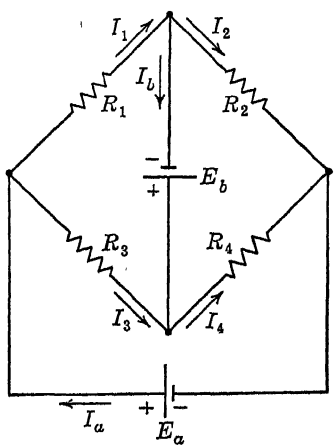


FIG. 29-5. Bridge circuit.

For the bridge circuit of Fig 29-5 Kirchhoff's Laws yield

$$\begin{aligned} R_1I_1 + R_2I_2 - R_4I_4 - R_3I_3 &= 0; \\ R_1I_1 - R_3I_3 &= E_b; \\ R_1I_1 + R_2I_2 &= E_a; \\ I_a - I_1 - I_3 &= 0; \\ I_1 - I_2 - I_b &= 0; \\ I_3 + I_b - I_4 &= 0. \end{aligned}$$

Show from the above equations that

$$I_b = \frac{(R_2R_3 - R_4R_1)E_a + (R_1 + R_2)(R_3 + R_4)E_b}{R_1R_2R_3 + R_2R_3R_4 + R_3R_4R_1 + R_4R_1R_2}.$$

A Wheatstone bridge results upon replacing the generator E_b of Fig. 29-5 by a galvanometer. Upon replacing E_b by $-RI_b$ in the system of equations for the bridge (R representing the galvanometer resistance), show that I_b becomes

$$I_b = \frac{(R_2R_3 - R_4R_1)E_a}{R_1R_2R_3 + R_2R_3R_4 + R_3R_4R_1 + R_4R_1R_2 + R(R_1 + R_2)(R_3 + R_4)}.$$

CHAPTER 30

GRAPHS

30–1. Graphs in Rectangular Coordinates. In Figs. 30–2 through 30–16 are presented for reference a collection of useful graphs in rectangular coordinates.

The *conic sections* shown in Figs. 30–3 through 30–7 are so named because each arises on cutting a right circular cone by a plane. In Fig. 30–1 the conic sections are (a) ellipse, (b) hyperbola, and (c) parabola. Any equation of the type

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be shown to represent a conic section and, in particular, an ellipse, hyperbola, or parabola in accordance with whether $B^2 - 4AC$ is negative, positive, or zero, respectively.

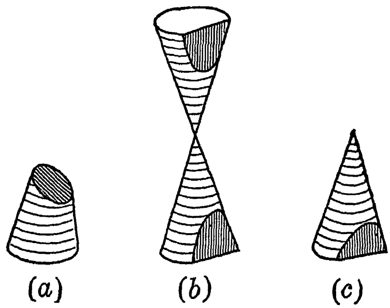


FIG. 30–1. Conic sections. (a) ellipse, (b) hyperbola, (c) parabola.

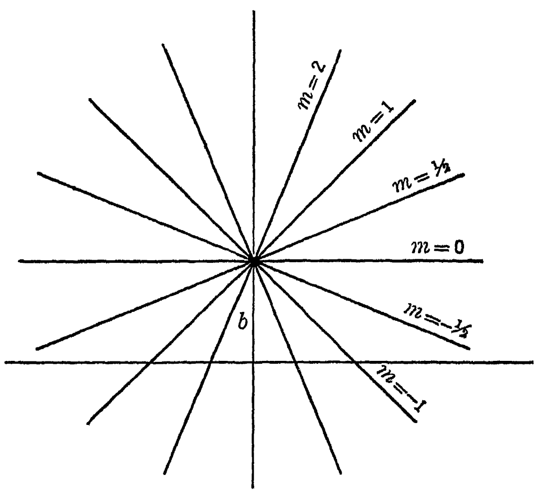


FIG. 30–2. Lines, $y = mx + b$.

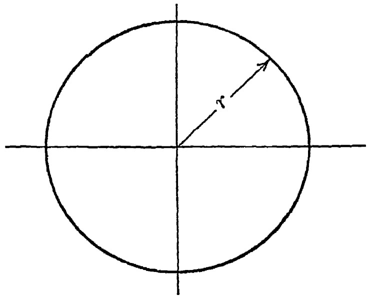


FIG. 30–3. Circle, $x^2 + y^2 = r^2$.

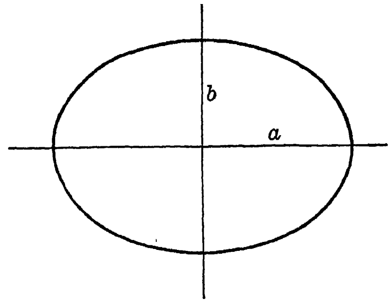


FIG. 30–4. Ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

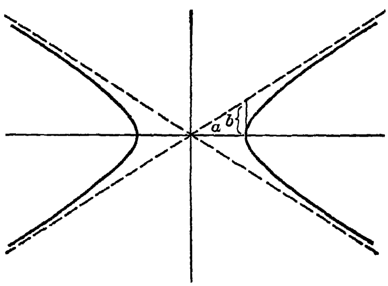


FIG. 30–5. Hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

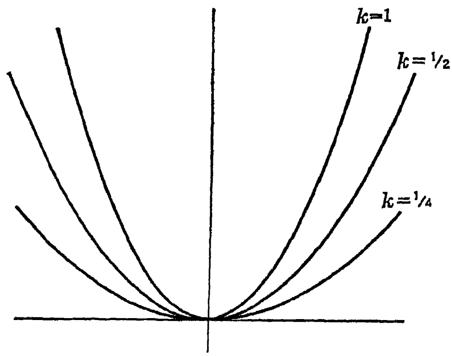


FIG. 30–6. Parabolas, $y = kx^2$.

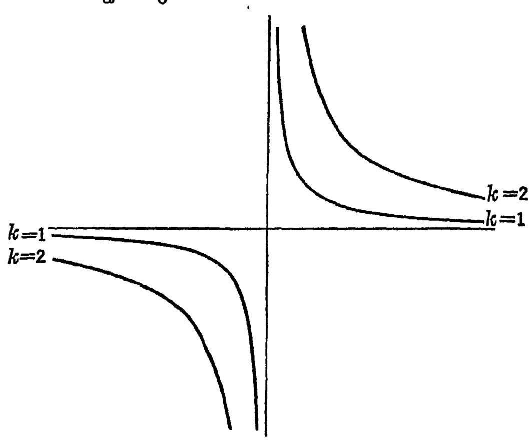
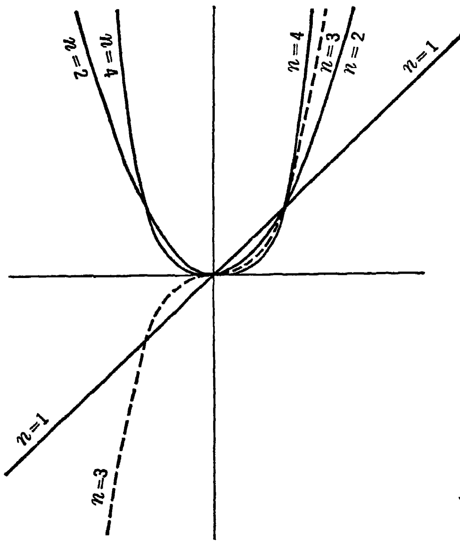


FIG. 30–7. Rectangular hyperbolas, $xy = k^2$.



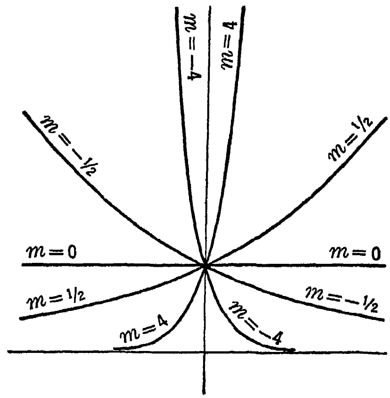


FIG. 30–8. Curves of power functions, $y = x^n$.

FIG. 30–9. Exponentials, $y = \epsilon^{mx}$

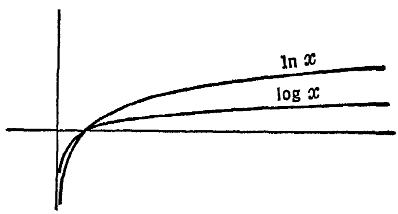


FIG. 30–10. Logarithmic curves, $y = \ln x$ and $y = \log x$.

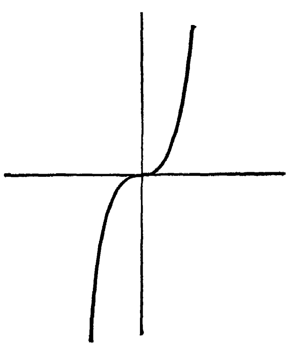


FIG. 30–11. Cubical parabola, $y = ax^3$.

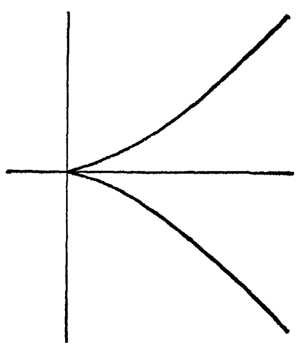


FIG. 30–12. Semi-cubical parabola, $y^2 = ax^3$.

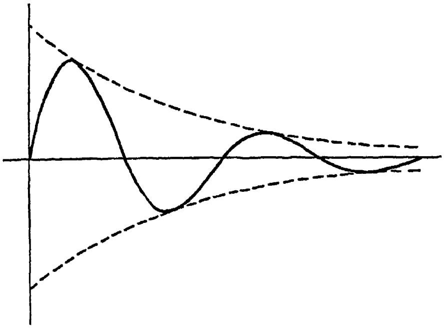


FIG. 30–13. Damped oscillation, $y = \epsilon^{-ax} \sin bx$.

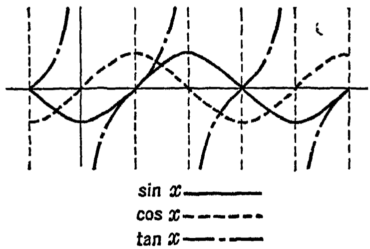


FIG. 30–14. Curves of the trigonometric functions sine, cosine, and tangent.

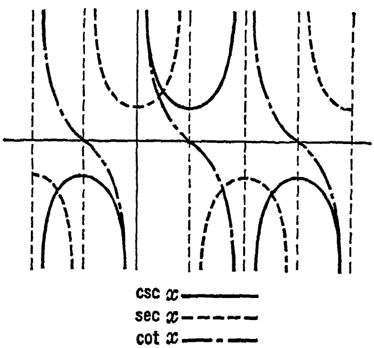


FIG. 30–15. Curves of the trigonometric functions cosecant, secant, and cotangent.

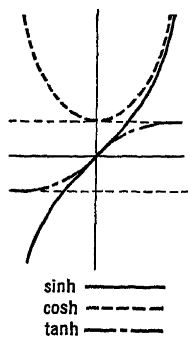


FIG. 30–16. Curves of the hyperbolic sine, cosine, and tangent.

Exercise 30–1

1. Identify (do not plot) the type of graph associated with each of the following equations:

   a. $x^2 + 2y^2 = 5$;

   b. $x^2 + 2y = 5$;

   c. $3x^2 - 2xy + 7 = 0$;

   d. $4x^2 - y^2 = 10$.

2. Determine the coordinates of the points of intersection of the curves of:

   a. Fig. 30–9;

   b. Fig. 30–10.

3. Sketch a curve of any general shape, and label it $y = f(x)$. Using the curve you have sketched as representative of the function $y = f(x)$, sketch curves of:

a. $y = f(x) + 2$;
b. $y = f(x - 3)$;
c. $y - k = f(x)$;
d. $y = f(x - h)$.

4. Based on your observations in Problem 3, together with the fact that $x^2 + y^2 = 9$ is a circle about the origin of radius 3, construct without plotting individual points curves of:

a. $(x - 2)^2 + y^2 = 9$;
b. $x^2 + (y - 2)^2 = 9$;
c. $(x - 2)^2 + (y + 3)^2 = 9$.

5. Corresponding to the curve of $y = f(x)$ in Problem 3 above, sketch curves of:

a. $y = -f(x)$;
b. $x = f(y)$.

6. With reference to Fig. 30-6 sketch curves of

a. $y = -x^2$;
b. $y^2 = x$;
c. $y^2 = -x$.

30-2. Graphs in Polar Coordinates. The same system used to describe a vector in polar form can be used to describe the position of a point in a plane. Thus, the point P of Fig. 30-17 is equally well described by the coordinates (r,θ) as by the coordinates (x,y). A system of coordinates which locates a point through an angle and a radius vector is called a *polar coordinate system.*

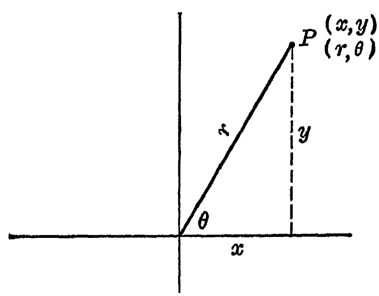


FIG. 30-17. Location of a point by rectangular coordinates and by polar coordinates.

A circle of radius a about the origin which is described by the equation $x^2 + y^2 = r^2$ in rectangular coordinates is in polar coordinates given by the equation $\rho = r$. A transformation

from rectangular to polar coordinates, and vice versa, is accomplished through the relations

$$x = \rho \cos \theta;\ y = \rho \sin \theta;$$

$$\rho = \sqrt{x^2 + y^2};\ \theta = \tan^{-1} \frac{y}{x}.$$

Polar coordinates are found to be convenient in studies involving directional effects, for example, in studies of directional characteristics of microphones, speakers, and antennas. Fig. 30-18 is an illustration of this usage.

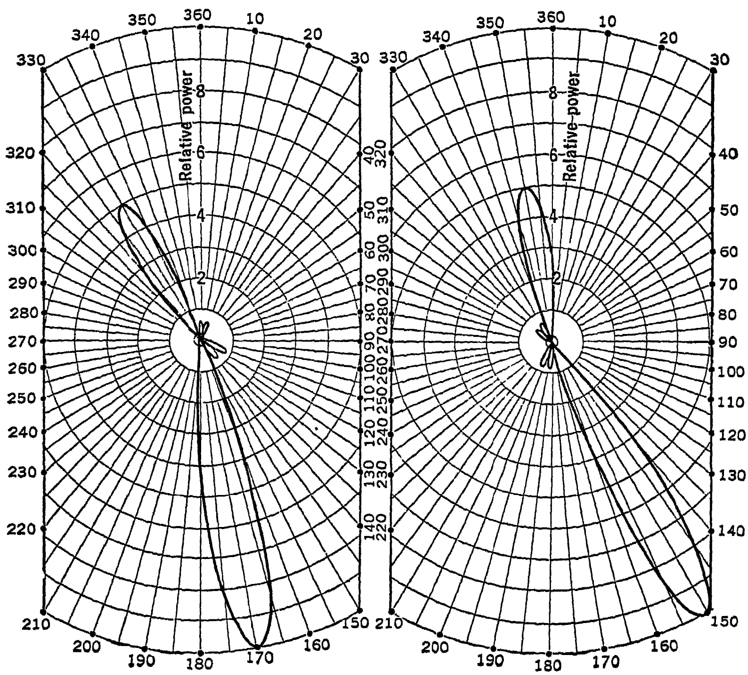


FIG. 30-18. Radiation patterns from two antenna systems of NBC's international broadcasting stations at Bound Brook, New Jersey. The 150° beam is aimed at Rio de Janeiro for the Portuguese speaking regions of South America; the 170° beam is directed at the center of the Spanish speaking countries in South America. (Graphs reproduced from the *RCA Review* of July, 1941.)

Miscellaneous graphs in polar form are shown in Figures 30-19 through 30-24.

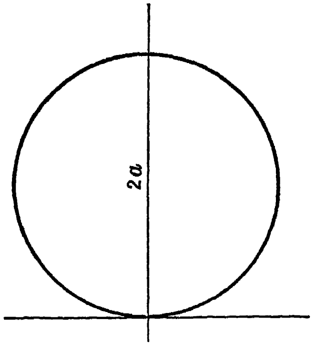


FIG. 30-19. Circle, $\rho = 2a \sin \theta$.

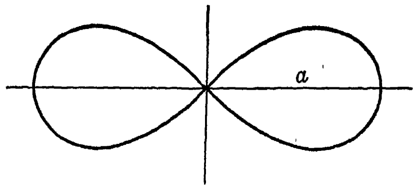


FIG. 30-20. Lemniscate, $\rho^2 = a^2 \cos 2\theta$.

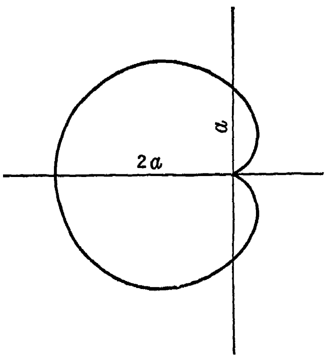


FIG. 30-21. Cardioid, $\rho = a(1 - \cos \theta)$.

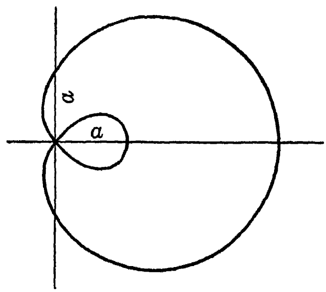


FIG. 30-22. Limacon, $\rho = a(1 + 2 \cos \theta)$.

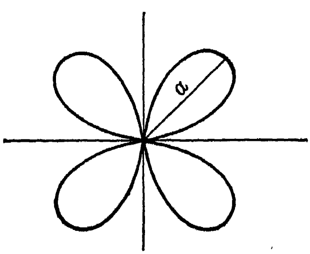


FIG. 30-23. Four-leaf rose, $\rho = a \sin 2\theta$.

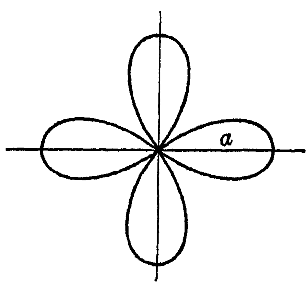


FIG. 30-24. Four-leaf rose, $\rho = a \cos 2\theta$

Exercise 30-2

1. Identify (do not plot) the type of graph associated with each of the following equations:

   a. $\rho = \sin\theta$.
   b. $\rho = 5 - 5\cos\theta$.
   c. $\rho = 100\sin 2\theta$.

2. Transform to polar coordinates:

   a. $x = k$,
   b. $y = c$,
   c. $y = x$,
   d. $xy = 1$.

3. Show that the equation of the lemniscate, $\rho^2 = a^2\cos 2\theta$, in rectangular coordinates is given by $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

4. Show that the equation of the cardioid, $\rho = a(1 - \cos\theta)$, in rectangular coordinates is given by

$$x^2 + y^2 + ax = a\sqrt{x^2 + y^2}.$$

5. Determine the coordinates (ρ,θ) of the farthest right intersection with the x-axis of the limacon of Fig. 30-22.

30-3. Parametric Equations. For some curves it is simplest to describe the relations involved between x and y in terms of a third variable, or parameter. This requires two equations in the three variables instead of just one equation in x and y. The two equations are referred to as *parametric equations.* The parametric equations of a circle are

$$x = r\cos\theta,$$

and

$$y = r\sin\theta.$$

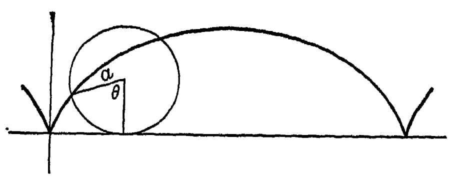


FIG. 30-25. Cycloid, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

On eliminating θ between these two equations, we obtain

$$x^2 + y^2 = r^2,$$

In Figs. 30-25 and 30-26 are shown curves which are useful in describing the electron paths in a magnetron. These curves are generated by tracing the course of a particular point on a disc which rolls along the x-axis. The corresponding equations are most simply expressed in parametric form.

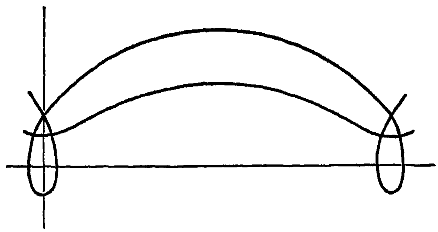


FIG. 30-26. Trochoids, $x = a\theta - b \sin \theta$, $y = a - b \cos \theta$.

Exercise 30-3

Show that the parametric equations of the cycloid of Fig. 30-25 are $x = a\ (\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

30-4. Nomographs. A *nomograph* is an alignment chart used in engineering practice to facilitate the solution of some frequently used equation. A common form of nomograph consists of three parallel lines, each line being graduated with a scale of values corresponding to one variable in the equation for which the graph is designed. The lines are so graduated and spaced that a straight edge which is placed through points on two of the lines will intersect the third line at the position which corresponds to a solution of the equation. The nomograph of Fig. 30-27 is designed for use with the commonly occurring equation for resonant circuits:

$$f = \frac{1}{2\pi\sqrt{LC}}.$$

If any two quantities in the above equation are given, the third is found to lie on a line which joins the first two.

Example. What capacitance is required to tune a resonant circuit to 5 megacycles with an inductance of 20 microhenrys? On the nomograph of Fig. 30-27 a straight edge joining 20 on the left scale and 5 on the center scale intersects the right scale at 50. Thus a capacitance of 50 micromicrofarads is required.

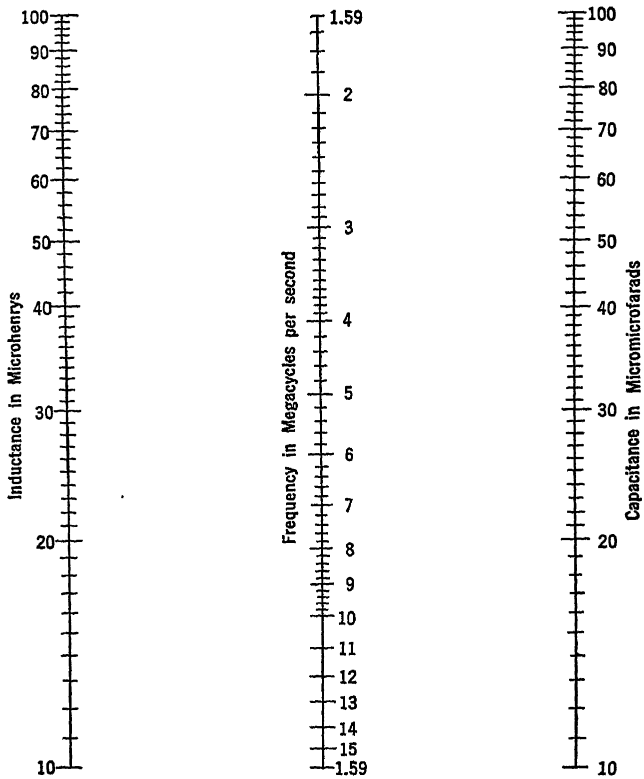


FIG. 30-27. Nomograph relating inductance, capacitance, and frequency in a resonant circuit.

Exercise 30-4

1. Referring to the equation $f = \dfrac{1}{2\pi\sqrt{LC}}$, show that the range of the nomograph of Fig. 30-27 may be extended by multiplying by the same number, the scale values on each of the vertical lines. This scheme of extending the range does not apply to nomographs in general, but for any particular nomograph a method of extension can be devised from a consideration of the equation.

2. Using the nomograph of Fig. 30-27, find the resonant frequency corresponding to a capacitance of 5 micromicrofarads and an inductance of 40 microhenrys.

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